Enrichment for the Bicategory of Orbispaces

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Topological Groupoids

Objects $\mathcal{G} = \left(\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \xrightarrow{m} \mathcal{G}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathcal{G}_0 \right)$ with $u: \mathcal{G}_0 \to \mathcal{G}_1$

Homs For topological groupoids $\mathcal{G}$ and $\mathcal{H}$,

$$\mathrm{Hom}(\mathcal{G}, \mathcal{H})$$

has the structure of a topological groupoid. We write:

$$\mathcal{G}\mathrm{Map}(\mathcal{G}, \mathcal{H}).$$

$$\mathcal{G}\mathrm{Map}(\mathcal{G},\mathcal{H})_0 \rightarrowtail \mathrm{Map}(\mathcal{G}_0,\mathcal{H}_0)\times \mathrm{Map}(\mathcal{G}_1,\mathcal{H}_1)$$

$$\begin{tikzcd}
\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \arrow[r, "m"] \arrow[d, "\varphi_1\times\varphi_1"'] \arrow[dr, phantom, "/\!/"] & \mathcal{G}_1 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] \arrow[d, "\varphi_1"] \arrow[dr, phantom, "/\!/"] & \mathcal{G}_0 \arrow[l, "u"'] \arrow[d, "\varphi_0"] \\
\mathcal{H}_1 \times_{\mathcal{H}_0} \mathcal{H}_1 \arrow[r, "m"'] & \mathcal{H}_1 \arrow[r, shift left=3, "s"] \arrow[r, "u"] & \mathcal{H}_0 \arrow[l, shift left=3, "t"]
\end{tikzcd}$$

$$\mathcal{G}\mathrm{Map}(\mathcal{G},\mathcal{H})_1 \rightarrowtail \mathrm{Map}(\mathcal{G}_0,\mathcal{H}_1)\times \mathcal{G}\mathrm{Map}(\mathcal{G},\mathcal{H})_0\times \mathcal{G}\mathrm{Map}(\mathcal{G},\mathcal{H})_0 .$$

$$\begin{tikzcd}
 & \mathcal{H}_1 \arrow[d, shift right=1, "s"'] \arrow[d, shift left=1, "t"] \\
\mathcal{G}_0 \arrow[ur, "\alpha"] \arrow[r, shift left=1, "\varphi_0"] \arrow[r, shift right=1, "\psi_0"'] & \mathcal{H}_0
\end{tikzcd}$$

$s\alpha = \varphi_0$

$t\alpha = \psi_0$

$+$ naturality.

Enrichment

1. $$\mathcal{G}\mathrm{Map}(\mathcal{G},\mathcal{H}) \times \mathcal{G}\mathrm{Map}(\mathcal{H},\mathcal{K}) \xrightarrow{\circ} \mathcal{G}\mathrm{Map}(\mathcal{G},\mathcal{K})$$

is a homomorphism of topological groupoids.

2. $$\mathbb{I} \xrightarrow[i_{\mathcal{G}}]{} \mathcal{G}\mathrm{Map}(\mathcal{G},\mathcal{G})$$
$$* \longmapsto 1_{\mathcal{G}} : \mathcal{G} \to \mathcal{G}$$

3. Strictly associative and unital.

Question

Is the bicategory of fractions

$$\mathrm{ProperEtaleGpds}(\mathcal{E}^{-1})$$

enriched in topological groupoids?

The Space $\mathrm{OMap}(\mathcal{G}, \mathcal{H})_0$

1. This space encodes

$$\begin{tikzcd} & \mathcal{G}^*\mathcal{U} \arrow[dl, "j_{\mathcal{U}}"'] \arrow[dr, "\varphi"] & \\ \mathcal{G} & & \mathcal{H} \end{tikzcd}$$

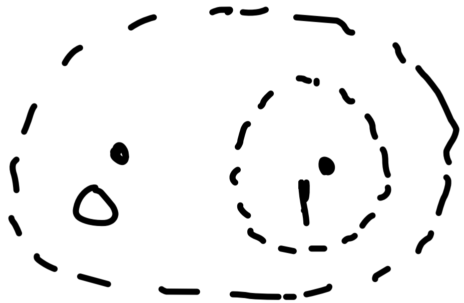
Where $\mathcal{U}$ is a non-repeating collection of opens in $\mathcal{G}_0$ that meets every orbit of $\mathcal{G}$.

The Space $\mathrm{OMap}(\mathcal{G}, \mathcal{H})_0$

2. There is a 1-1 correspondence

$$\frac{U \subseteq \mathcal{G}_0 \text{ open}}{\chi_U : \mathcal{G}_0 \longrightarrow \mathbb{S}}$$

Where $\mathbb{S} \equiv$ Sierpinski space



$$\chi_U(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

The Space $\mathrm{OMap}(\mathcal{G}, \mathcal{H})_0$

3. A collection $\mathcal{U}$ of $|\mathcal{U}|$ open subsets of $\mathcal{G}_0$ would be classified by

$$\chi_{\mathcal{U}} : \mathcal{G}_0 \longrightarrow \mathbb{S}^{|\mathcal{U}|}$$

4. We only want to consider $\mathcal{U}$'s that are essential coverings:

   - no repeats

   - meeting every orbit

5. We want to use the same cardinality for all covers:

- allow $\emptyset$ (repeated as needed) as part of the cover;
- use $|\mathcal{T}(\mathcal{G}_0)|$, the cardality of the topology of $\mathcal{G}_0$.

6. We write

$$EC(G)\text{-map}(\mathcal{G}_0, \mathcal{S}^{|\mathcal{T}(\mathcal{G}_0)|}) \rightarrowtail \mathrm{Map}(\mathcal{G}_0, \mathcal{S}^{|\mathcal{T}(\mathcal{G}_0)|})$$

for the subspace classifying essential covers.

The Space $\mathrm{OMap}(\mathcal{G},\mathcal{H})_0$

5. The space of generalized maps

$$\begin{tikzcd} & \mathcal{G}^*\mathcal{U} \arrow[dl, "j_{\mathcal{U}}"'] \arrow[dr, "\varphi"] & \\ \mathcal{G} & & \mathcal{H} \end{tikzcd}$$

is given by the pullback

$$\begin{tikzcd} \mathrm{OMap}(\mathcal{G},\mathcal{H})_0 \arrow[r] \arrow[d] & \bigcup_{\mathcal{U}} \mathcal{G}\mathrm{map}(\mathcal{G}^*\mathcal{U},\mathcal{H}) \arrow[d] \\ EC(\mathcal{G})\text{-}\mathrm{map}\left(\mathcal{G}_0, \mathcal{S}^{|\mathcal{T}(\mathcal{G}_0)|}\right) \arrow[r] & \left\{\begin{matrix}\text{ess. covers} \\ \text{of } \mathcal{G}_0\end{matrix}\right\}^{\text{indisc.}} \end{tikzcd}$$

Topology on $\bigcup_{\mathcal{U}} \mathcal{G}\,\mathrm{Map}(\mathcal{G}^*\mathcal{U}, \mathcal{H})$

- Subbasis sets: $\langle C, D, V, W \rangle$

where $C \subseteq \mathcal{G}_0$, $D \subseteq \mathcal{G}_1$ are compact, and $V \subseteq \mathcal{H}_0$ and $W \subseteq \mathcal{H}_1$ are open.

- $\langle C, V \rangle = \{ \varphi : \mathcal{G}^*\mathcal{U} \to \mathcal{H} \mid \exists\, C' \subseteq \mathcal{G}^*\mathcal{U}_0,$
$D' \subseteq \mathcal{G}^*\mathcal{U}_1$ compact subsets s.t. $j_{\mathcal{U}}(C') = C,$
$\varphi_0(C') \subseteq V,\ j_{\mathcal{U}}(D') = D,\ \varphi_1(D') \subseteq W \}$

The Space $\mathcal{O}\mathrm{Map}(\mathcal{G},\mathcal{H})_1$

To obtain a description of the space of arrows,

$$\mathcal{O}\mathrm{Map}(\mathcal{G},\mathcal{H})_1$$

we need to revisit the definition of 2-cells in the bicategory of fractions.

Revisit the 2-cells

- Recall condition WBF4:

  For each
$$\begin{tikzcd} & \cdot \arrow[dr, "w"] & \\ \cdot \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & \cdot \\ & \cdot \arrow[ur, "w"'] & \end{tikzcd}$$
  there is
$$\begin{tikzcd} & \cdot \arrow[dr, "f"] & \\ \cdot \arrow[ur, "u"] \arrow[dr, "u"'] \arrow[rr, phantom, "\Downarrow\tilde{\alpha}"] & & \cdot \\ & \cdot \arrow[ur, "g"'] & \end{tikzcd}$$
  s.t. $\quad w\tilde{\alpha} = \alpha u$

- For essential equivalences we have additional results.

For essential equivalences we have:

Unit BFy $\forall$

$$\begin{tikzcd} \cdot \arrow[r, bend left, "\varphi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\Downarrow\alpha"] & \cdot \arrow[r, bend left, "\varepsilon"] \arrow[r, bend right, "\varepsilon"'] & \cdot \end{tikzcd} \qquad \exists! \quad \begin{tikzcd} \cdot \arrow[r, bend left, "\varphi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\Downarrow\tilde{\alpha}"] & \cdot \end{tikzcd} \quad \text{s.t.} \quad \varepsilon\tilde{\alpha} = \alpha$$

<u>Lemma</u>: For $\varepsilon : \mathcal{H}' \longrightarrow \mathcal{H}$ an essential equivalence,

$$\varepsilon^* : \mathcal{G}\mathrm{Map}(\mathcal{G}, \mathcal{H}') \longrightarrow \mathcal{G}\mathrm{Map}(\mathcal{G}, \mathcal{H})$$

is full and faithful.

If $\varepsilon_0 : \mathcal{H}'_0 \to \mathcal{H}_0$ is also open, then for any $\varphi, \psi : \mathcal{G} \rightrightarrows \mathcal{H}'$,

$$\mathcal{G}\mathrm{Map}(\mathcal{G}, \mathcal{H}')(\varphi, \psi) \cong \mathcal{G}\mathrm{Map}(\mathcal{G}, \mathcal{H})(\varepsilon\varphi, \varepsilon\psi)$$

is a homeomorphism.

There is also a co-version for essential equivalences:

Unit Co-BF4 $\forall$
$$\begin{tikzcd} \cdot \arrow[r, "\varepsilon"] & \cdot \arrow[r, "\varphi"] \arrow[d, phantom, "\Downarrow\alpha"] & \cdot \\ \cdot \arrow[r, "\varepsilon"'] & \cdot \arrow[r, "\psi"'] & \cdot \end{tikzcd}
\quad \exists! \quad
\begin{tikzcd} \cdot \arrow[r, bend left, "\varphi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "\Downarrow\bar{\alpha}"] & \cdot \end{tikzcd}
\quad \text{s.t. } \bar{\alpha}\varepsilon = \alpha$$

<u>Lemma:</u> Let $\varepsilon : \mathcal{G} \to \mathcal{G}'$ be an essential equivalence.

Then $\varepsilon^* : \mathcal{G}Map(\mathcal{G}', \mathcal{H}) \longrightarrow \mathcal{G}Map(\mathcal{G}, \mathcal{H})$ is full and faithful.

If ε_0 is open, then for any $\varphi, \psi : \mathcal{G}' \rightrightarrows \mathcal{H}$, ε^* gives rise to a homeomorphism,

$$\mathcal{G}Map(\mathcal{G}', \mathcal{H})(\varphi, \psi) \cong \mathcal{G}Map(\mathcal{G}, \mathcal{H})(\varphi\varepsilon, \psi\varepsilon)$$

Representatives for 2-Cells

Since the class of essential equivalences also satisfies 3-for-2, we obtain:

Given
$$\begin{tikzcd}
 & \bullet \arrow[dl, "u_1"'] \arrow[dr, "f_1"] & \\
\bullet \arrow[r, phantom, "\alpha"] & \bullet \arrow[u, "v_1"'] \arrow[d, "v_2"] \arrow[r, phantom, "\beta"] & \bullet \\
 & \bullet \arrow[ul, "u_2"] \arrow[ur, "f_2"'] &
\end{tikzcd}$$
and
$$\begin{tikzcd}
 & \bullet \arrow[dl, "u_1"'] \\
\bullet \arrow[r, phantom, "\alpha'"] & \bullet \arrow[u, "v_1'"'] \arrow[d, "v_2'"] \\
 & \bullet \arrow[ul, "u_2"]
\end{tikzcd}$$
there is a unique
$$\begin{tikzcd}
\bullet \arrow[dr, "f_1"] & \\
\bullet \arrow[u, "v_1'"] \arrow[d, "v_2'"'] \arrow[r, phantom, "\beta'"] & \bullet \\
\bullet \arrow[ur, "f_2"'] &
\end{tikzcd}$$
such that

$$\begin{tikzcd}
 & \bullet \arrow[dl, "u_1"'] \arrow[dr, "f_1"] & \\
\bullet \arrow[r, phantom, "\alpha"] & \bullet \arrow[u, "v_1"'] \arrow[d, "v_2"] \arrow[r, phantom, "\beta"] & \bullet \\
 & \bullet \arrow[ul, "u_2"] \arrow[ur, "f_2"'] &
\end{tikzcd}
\sim
\begin{tikzcd}
 & \bullet \arrow[dl, "u_1"'] \arrow[dr, "f_1"] & \\
\bullet \arrow[r, phantom, "\alpha'"] & \bullet \arrow[u, "v_1'"'] \arrow[d, "v_2'"] \arrow[r, phantom, "\beta'"] & \bullet \\
 & \bullet \arrow[ul, "u_2"] \arrow[ur, "f_2"'] &
\end{tikzcd}$$

Two types of nice choices:

1. for essential covering maps:

$$\begin{tikzcd}
 & \mathcal{G}^*(\mathcal{U}) \arrow[dl, bend right, "j_{\mathcal{U}}"'] \\
\mathcal{G} \arrow[r, phantom, "\cong"] & \mathcal{G}^*(\mathcal{W}_{\mathcal{U},\mathcal{V}}) \arrow[u, "j_1"'] \arrow[d, "j_2"] \\
 & \mathcal{G}^*(\mathcal{V}) \arrow[ul, bend left, "j_{\mathcal{V}}"]
\end{tikzcd}$$

essential common refinement

2. for essential equivalences:

$$\begin{tikzcd}
 & \mathcal{G}' \arrow[dl, bend right, "\varepsilon"'] \\
\mathcal{G} \arrow[r, phantom, "\rho_{\varepsilon,\varepsilon'}"] & \mathcal{P}_{\varepsilon,\varepsilon'} \arrow[u, "p"'] \arrow[d, "q"] \\
 & \mathcal{G}'' \arrow[ul, bend left, "\varepsilon'"]
\end{tikzcd}$$

pseudo pullback.

Pseudo Pullbacks and Essential Covering Maps

$$\begin{tikzcd}
& \mathcal{G}^*(\mathcal{U}) \arrow[dl, bend right, "j_{\mathcal{U}}"'] \\
\mathcal{G} \arrow[r, phantom, "\rho_{\mathcal{U},\mathcal{V}}"] & \mathcal{P}_{\mathcal{U},\mathcal{V}} \arrow[u, "p"'] \arrow[d, "q"] \\
& \mathcal{G}^*(\mathcal{V}) \arrow[ul, bend left, "j_{\mathcal{V}}"]
\end{tikzcd}$$

the composites $j_{\mathcal{U}}p$ and $j_{\mathcal{V}}q$ are not necessarily essential covering maps.

The functor

$$\mathrm{ProperEt.Gpds}(\mathcal{E}^{-1}) \hookrightarrow \mathrm{ProperEt.Gpds}(\mathcal{E}^{-1})$$

can be made a strict functor.

So we can use the 2-cells from $\mathrm{ProperEt.Gpds}(\mathcal{E}^{-1})$ in $\mathrm{ProperEt.Gpds}(\mathcal{E}^{-1})$.

Horizontal Whiskering, I

$$
\begin{tikzcd}
 & & & & \mathcal{H}^*V_1 \arrow[dll, bend right, "j_{V_1}"'] \arrow[drr, bend left, "\psi_1"] & & \\
\mathcal{G} & \mathcal{G}^*\mathcal{U} \arrow[l, "j_{\mathcal{U}}"'] \arrow[r, "\varphi"] & \mathcal{H} & \rho_{V_1,V_2} & P_{V_1,V_2} \arrow[u, "p"] \arrow[d, "q"] & \beta \Downarrow & \mathcal{K} \\
 & & & & \mathcal{H}^*V_2 \arrow[ull, bend left, "j_{V_2}"] \arrow[urr, bend right, "\psi_2"'] & &
\end{tikzcd}
$$

Step 1: Compose the arrows

$$
\begin{tikzcd}
 & & \mathcal{G}W_i \arrow[dl, "j_i"'] \arrow[dr, "\overline{\varphi}_i"] \arrow[ddll, bend right, "j_{W_i}"'] & & \\
 & \mathcal{G}^*\mathcal{U} \arrow[dl, "j_{\mathcal{U}}"'] \arrow[dr, "\varphi"'] & \gamma_i & \mathcal{H}^*V_i \arrow[dl, "j_{V_i}"] \arrow[dr, "\psi_i"] & \\
\mathcal{G} & & \mathcal{H} & & \mathcal{K}
\end{tikzcd}
$$

for $i = 1, 2$

Horizontal Whiskering, I

$$\begin{tikzcd}
& \mathcal{G}^*\mathcal{W}_1 \arrow[dl, "j_{\mathcal{W}_1}"'] \arrow[r, "\overline{\varphi}_1"] & \mathcal{H}^*\mathcal{V}_1 \arrow[dr, bend left, "\psi_1"] & \\
\mathcal{G} & \mathcal{P}_{\mathcal{W}_1,\mathcal{W}_2} \arrow[l, "\rho_{\mathcal{W}_1,\mathcal{W}_2}"'] \arrow[u, "p'"'] \arrow[d, "q'"] \arrow[r, "\theta", "!"'] & \mathcal{P}_{\mathcal{V}_1,\mathcal{V}_2} \arrow[u, "p"'] \arrow[d, "q"] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{K} \\
& \mathcal{G}^*\mathcal{W}_2 \arrow[ul, bend left] \arrow[r, "\overline{\varphi}_2"'] & \mathcal{H}^*\mathcal{V}_2 \arrow[ur, bend right, "\psi_2"'] &
\end{tikzcd}$$

With θ induced by:

$$\begin{tikzcd}
& \mathcal{G}^*\mathcal{W}_1 \arrow[r, "\overline{\varphi}_1"] \arrow[dr, "j_1"] & \mathcal{H}^*\mathcal{V}_1 \arrow[dr, "j_{\mathcal{V}_1}"] \arrow[d, phantom, "\gamma_1"] & \\
\mathcal{P}_{\mathcal{W}_1,\mathcal{W}_2} \arrow[ur, "p'"] \arrow[dr, "q'"'] \arrow[rr, phantom, "\tilde{\rho}_{\mathcal{W}_1,\mathcal{W}_2}"] & & \mathcal{G}^*\mathcal{U} \arrow[r, "\varphi"'] \arrow[d, phantom, "\gamma_2"] & \mathcal{H} \\
& \mathcal{G}^*\mathcal{W}_2 \arrow[ur, "j_2"] \arrow[r, "\overline{\varphi}_2"'] & \mathcal{H}^*\mathcal{V}_2 \arrow[ur, "j_{\mathcal{V}_2}"'] &
\end{tikzcd}$$

Horizontal Whiskering, II

$$\begin{tikzcd}
 & \mathcal{G}^*\mathcal{U}_1 \arrow[dl, "j_{\mathcal{U}_1}"'] \arrow[dr, bend left, "\varphi_1"] & & & \\
\mathcal{G} \arrow[r, phantom, "\rho_{\mathcal{U}_1,\mathcal{U}_2}"] & \mathcal{P}_{\mathcal{U}_1,\mathcal{U}_2} \arrow[u, "p"] \arrow[d, "q"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{H} & \mathcal{H}^*\mathcal{V} \arrow[l, "j_{\mathcal{V}}"'] \arrow[r, "\psi"] & \mathcal{K} \\
 & \mathcal{G}^*\mathcal{U}_2 \arrow[ul, "j_{\mathcal{U}_2}"] \arrow[ur, bend right, "\varphi_2"'] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \mathcal{G}^*\mathcal{W}_i \arrow[dl, "j_i"'] \arrow[dr, "\bar{\varphi}_i"] \arrow[ddll, bend right, "j_{\mathcal{W}_i}"'] \arrow[dd, phantom, "\gamma_i"] & & \\
 & \mathcal{G}^*\mathcal{U}_i \arrow[dl, "j_{\mathcal{U}_i}"'] \arrow[dr, "\varphi_i"'] & & \mathcal{H}^*\mathcal{V} \arrow[dl, "j_{\mathcal{V}}"] \arrow[dr, "\psi"] & \\
\mathcal{G} & & \mathcal{H} & & \mathcal{K}
\end{tikzcd}$$

Horizontal Whiskering, II

Let $\tilde{\alpha}$ be the lifting of

$$
\begin{tikzcd}
\mathcal{G}^*_{W_1} \arrow[r, "\gamma_1", "j_1"'] \arrow[rrd, bend left=40, "j_v \overline{\varphi_1}"] & \mathcal{G}^*_{U_1} \arrow[rd, "\varphi_1"] & \\
\mathcal{P}_{W_1,W_2} \arrow[u, "p'"] \arrow[d, "g'"'] \arrow[r, "\Theta"] \arrow[ur, phantom, "="] \arrow[dr, phantom, "="] & \mathcal{P}_{U_1,U_2} \arrow[u, "p"'] \arrow[d, "g"] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{H} \\
\mathcal{G}^*_{W_2} \arrow[r, "j_2", "\gamma_2"'] \arrow[rru, bend right=40, "j_v \overline{\varphi_2}"'] & \mathcal{G}^*_{U_2} \arrow[ru, "\varphi_2"'] &
\end{tikzcd}
$$

with respect to j_v.

Horizontal Whiskering, II

Then the resulting 2-cell is:

$$
\begin{tikzcd}
 & \mathcal{G}^*\mathcal{W}_1 \arrow[dl, bend right, "j_{\mathcal{W}_1}"'] \arrow[dr, bend left, "\psi\bar{\varphi}_1"] & \\
\mathcal{G} \arrow[r, phantom, "\rho_{\mathcal{W}_1,\mathcal{W}_2}"] & \mathcal{P}_{\mathcal{W}_1,\mathcal{W}_2} \arrow[u, "p'"'] \arrow[d, "q'"] \arrow[r, phantom, "\psi\tilde{\alpha}\ \Downarrow"] & \mathcal{K} \\
 & \mathcal{G}^*\mathcal{W}_2 \arrow[ul, bend left, "j_{\mathcal{W}_2}"] \arrow[ur, bend right, "\psi\bar{\varphi}_2"'] &
\end{tikzcd}
$$

The Space $\mathcal{G}\mathrm{Map}(\mathcal{G},\mathcal{H})_1$

The right hand side of

$$\begin{tikzcd}
 & \mathcal{G}^*\mathcal{U} \arrow[dl, "j_\mathcal{U}"'] \arrow[dr, "\varphi"] & \\
\mathcal{G} \arrow[r, phantom, "P_{\mathcal{U},\mathcal{V}}"] & P_{\mathcal{U},\mathcal{V}} \arrow[u, "\pi_1"] \arrow[d, "\pi_3"] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{H} \\
 & \mathcal{G}^*\mathcal{V} \arrow[ul, "j_\mathcal{V}"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

is a point of:

$$Q_{\mathcal{U},\mathcal{V}} = \mathcal{G}\mathrm{Map}(\mathcal{G}^*\mathcal{U},\mathcal{H})_0 \underset{\pi_1^*,\,\mathcal{G}\mathrm{Map}(P_{\mathcal{U},\mathcal{V}},\mathcal{H})_0,\,s}{\times} \mathcal{G}\mathrm{Map}(P_{\mathcal{U},\mathcal{V}},\mathcal{H})_1 \underset{t,\,\mathcal{G}\mathrm{Map}(P_{\mathcal{U},\mathcal{V}},\mathcal{H})_0,\,\pi_3^*}{\times} \mathcal{G}\mathrm{Map}(\mathcal{G}^*\mathcal{V},\mathcal{H})_0$$

The Space $\mathrm{OMap}(\mathcal{G},\mathcal{H})$,

So the space of all diagrams

$$\begin{tikzcd}
 & \mathcal{G}^*\mathcal{U} \arrow[dl, bend right, "j_{\mathcal{U}}"'] \arrow[dr, bend left, "\varphi"] & \\
\mathcal{G} \arrow[r, phantom, "\rho_{\mathcal{U},\mathcal{V}}"] & P_{\mathcal{U},\mathcal{V}} \arrow[u] \arrow[d] \arrow[r, phantom, "\beta"] & \mathcal{H} \\
 & \mathcal{G}^*\mathcal{V} \arrow[ul, bend left, "j_{\mathcal{V}}"] \arrow[ur, bend right, "\psi"'] &
\end{tikzcd}$$

is the pullback,

$$\begin{tikzcd}
\mathrm{OMap}(\mathcal{G},\mathcal{H}), \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \bigcup\limits_{\mathcal{U},\mathcal{V}} Q_{\mathcal{U},\mathcal{V}} \arrow[d] \\
EC(\mathcal{G})\text{-}\mathrm{map}(\mathcal{G}_0, \$^{|J(\mathcal{G}_0)|}) \times EC(\mathcal{G})\text{-}\mathrm{map}(\mathcal{G}_0, \$^{|J(\mathcal{G}_0)|}) \arrow[r] & \left\{ (\mathcal{U},\mathcal{V}) \,;\, \text{ess. covers of } \mathcal{G}_0 \right\}
\end{tikzcd}$$

Composition

__Theorem__ Composing by $\mathcal{G} \xleftarrow{j_{\mathcal{U}}} \mathcal{G}^*\mathcal{U} \xrightarrow{\varphi} \mathcal{H}$ induces (continuous) homomorphisms of topological groupoids:

$$(j_{\mathcal{U}}, \varphi)^* : \mathrm{OMap}(\mathcal{H}, \mathcal{K}) \longrightarrow \mathrm{OMap}(\mathcal{G}, \mathcal{K})$$

$$(j_{\mathcal{U}}, \varphi)_* : \mathrm{OMap}(\mathcal{L}, \mathcal{G}) \longrightarrow \mathrm{OMap}(\mathcal{L}, \mathcal{H})$$

Enrichment and Morita Invariance

Corollary 1: ProperEtGrpds$(\mathcal{C}^{-1})$ is enriched over the 2-category of (Hausdorff) topological groupoids.

Corollary 2: If $\mathcal{G} \sim_m \mathcal{G}'$ and $\mathcal{H} \sim \mathcal{H}'$, then

$$\mathrm{OMap}(\mathcal{G}, \mathcal{H}) \sim_m \mathrm{OMap}(\mathcal{G}', \mathcal{H}')$$

Further Results

- When $\mathcal{G}$ has a compact orbit space, we may restrict ourselves to finite covers with compact closures.

- This results in a proper étale groupoid $\mathrm{OMap}_c(\mathcal{G}, \mathcal{H})$ with the property that

$$\mathrm{ProperEtGrpds}(\mathcal{E}^{-1})(\mathcal{L}, \mathrm{OMap}_c(\mathcal{G}, \mathcal{H})) \simeq \mathrm{ProperEtGrpds}(\mathcal{E}^{-1})(\mathcal{L} \times \mathcal{G}, \mathcal{H}).$$