

# Curvature and torsion without negatives

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# Overview

- Tangent categories provide an abstract framework for unifying many disparate notions of “derivative” and “tangent bundle”.
- Examples include smooth manifolds, SDG, schemes, Cartesian differential categories, Abelian functor calculus, potentially Goodwillie functor calculus (perhaps a 2 or infinity tangent category), tropical geometry...
- To encompass a variety of different examples, tangent categories do not assume one can negate tangent vectors.
- Many aspects of differential geometry have been developed in this setting: vector bundles, connections, differential forms, de Rham cohomology, vector fields, flows, Lie brackets...

# Overview

- However, some of these definitions have required assuming the existence of negatives, meaning they won't apply to all examples.
- One example has been curvature and torsion of a connection. For example, the standard definitions (for a covariant derivative on a smooth manifold) use negatives:

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u, v]} w$$

$$T(x, y) = \nabla_x y - \nabla_y x - [x, y]$$

- In this talk, we'll recall how to define curvature and torsion of a connection on an object in a tangent category, and then see how to re-work the definition so that negatives are not required.

# Tangent category definition

Definition (Rosický 1984, modified Cockett/Crutwell 2013)

A **tangent category** consists of a category  $\mathbb{X}$  with:

- **tangent bundle functor**: an endofunctor  $T : \mathbb{X} \rightarrow \mathbb{X}$ ;
- **projection of tangent vectors**: a natural transformation  $p : T \rightarrow 1_{\mathbb{X}}$ ;
- for each  $M$ , the pullback of  $n$  copies of  $p_M$  along itself exists; call this pullback  $T_n M$  (the “space of  $n$  tangent vectors at a point”)
- **addition and zero tangent vectors**: for each  $M \in \mathbb{X}$ ,  $p_M$  has the structure of a commutative monoid in the slice category  $\mathbb{X}/M$ ;

# Tangent category definition (continued)

## Definition

- **symmetry of mixed partial derivatives:** a natural transformation  $c : T^2 \rightarrow T^2$ ;
- **linearity of the derivative:** a natural transformation  $\ell : T \rightarrow T^2$ ;
- “the vertical bundle of the tangent bundle is trivial”;
- various coherence equations for  $\ell$  and  $c$ .

Say that tangent category *has negatives* if the monoid structure of each  $p_M : TM \rightarrow M$  is actually a group.

# Examples

- ❶ Finite dimensional smooth manifolds with the usual tangent bundle.
- ❷ Convenient manifolds with the kinematic tangent bundle.
- ❸ Any Cartesian differential category (includes all Fermat theories by a result of MacAdam, and Abelian functor calculus by a result of Bauer et. al.).
- ❹ The microlinear objects in a model of synthetic differential geometry (SDG).
- ❺ Commutative  $\text{ri}(n)$ gs and its opposite, as well as various other categories in algebraic geometry.
- ❻ The category of  $C^\infty$ -rings.
- ❼ With additional pullback assumptions, tangent categories are closed under slicing.

**Note:** Building on work of Leung, Garner has shown how tangent categories are a type of enriched category.

# Intuitive idea of a connection

**Idea:** a **connection** on a “bundle”  $q : E \rightarrow M$  is a choice of a horizontal and vertical co-ordinate system for  $TE$  (see diagram).

# Vertical bundle

## Definition

If  $q : E \rightarrow M$  is a bundle, its **vertical bundle**,  $V(E)$ , is the following pullback:

$$\begin{array}{ccc} V(E) & \xrightarrow{i} & T(E) \\ \downarrow & & \downarrow T(q) \\ M & \xrightarrow{0} & T(M) \end{array}$$



# Horizontal bundle

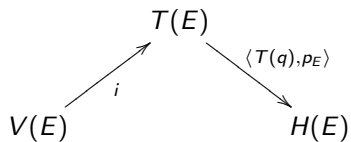
## Definition

If  $q : E \rightarrow M$  is a bundle, its **horizontal bundle**,  $H(E)$ , is the following pullback:

$$\begin{array}{ccc} H(E) & \longrightarrow & T(M) \\ \pi \downarrow & & \downarrow p_M \\ E & \xrightarrow{q} & M \end{array}$$

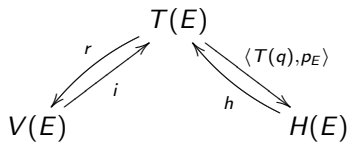
# Associated maps

A bundle then has the following diagram of maps:



# General connection

A **connection** on such a bundle is then required to have maps  $r, h$ :



satisfying various axioms.

# Connection on a vertically trivial bundle

- For vector bundles, the vertical bundle  $VE$  is trivial, in the sense that it is a fibred product:  $VE \cong E \times_M E$  (this is essentially how we *define* vector bundles in a tangent category).
- In this case, the vertical part of a connection is simply given by a map  $K : TE \rightarrow E$ .
- In particular, we axiomatically assume that the vector bundle of the tangent bundle is trivial, and so in this case the vertical part of a connection is given by a map  $T^2M \rightarrow TM$ ; the horizontal part is given by a map  $H : T_2M \rightarrow T^2M$ .
- We shall write  $(K, H)$  for a connection on the tangent bundle of  $M$ .

# Torsion

## Definition

A connection  $(K, H)$  on  $M$  is **torsion-free** if  $c_M K = K$ :

$$\begin{array}{ccc} T^2M & \xrightarrow{c_M} & T^2M \\ & \searrow K & \downarrow K \\ & & TM \end{array}$$

(Standard definition: for all  $x, y$ ,  $\nabla_x y - \nabla_y x - [x, y] = 0$ .)

## Definition

In a tangent category with negatives, the **torsion** of a connection is the difference

$$T^2M \xrightarrow{cK - K} TM.$$

# Curvature

## Definition

A connection  $(K, H)$  on  $M$  is **flat** (curvature-free) if  $c_{TM}T(K)K = T(K)K$ :

$$\begin{array}{ccccc}
 T^3M & \xrightarrow{c_{TM}} & T^3M & \xrightarrow{T(K)} & T^2M \\
 & \searrow & & & \downarrow K \\
 & & T^2M & \xrightarrow{K} & TM
 \end{array}$$

(Standard definition: for all  $u, v, w$ ,  $\nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w = 0$ .)

## Definition

In a tangent category with negatives, the **curvature** of a connection is the difference

$$T^3M \xrightarrow{cT(K)K - T(K)K} T^2M.$$

# Problems

There are several problems with these definitions:

- The torsion and curvature maps require negatives.
- Seems to be “higher-order” than the ordinary definitions (eg., torsion goes from  $T^2M$  instead of  $T_2M$ ).
- Neither definition uses  $H$ .

# Higher order?

- If these definitions really are higher-order, they should have more information than the standard definition. What is this extra information?
- However, when I actually did some calculations with what these notions told me for connections on simple smooth manifolds (eg., spheres), the higher-order terms always vanished!
- Actually, this holds more generally!



# Simplifying torsion

- Recall that if  $M$  has a connection  $K$ , every element of  $T^2M$  is uniquely given determined by its horizontal and vertical parts (see diagram).
- Thus, we can look at what the horizontal and vertical parts of the expression  $cK - K$  are.
- **The vertical parts vanish, and the horizontal part of  $K$  vanishes.** As a result, all the information in  $cK - K$  is contained in the expression

$$T_2M \xrightarrow{H} T^2M \xrightarrow{c_M} T^2M \xrightarrow{K} TM.$$

# New torsion definition

## Definition

For a connection  $(K, H)$  on  $M$ , its **torsion** is the map

$$T_2M \xrightarrow{H} T^2M \xrightarrow{c_M} T^2M \xrightarrow{K} TM$$

It is **torsion-free** if this is zero (that is, it equals  $\pi_0 p_0$ ).

- This solves all three previous problems simultaneously!
- I haven't seen anything quite like it in ordinary differential geometry.

# Simplifying curvature

- The curvature is a map out of  $T^3M$ : but with a connection, the splitting of  $T^2M$  also leads to a splitting of  $T^3M$ .
- Applying this splitting to the curvature expression  $cT(K)K - T(K)K$  shows that all its information is contained in the expression

$$\begin{array}{ccccccc}
 T_3M & \xrightarrow{\langle\langle\pi_0,\pi_1\rangle H, \langle\pi_0,\pi_2\rangle H\rangle} & T(T_2M) & \xrightarrow{T(H)} & T^3M & \xrightarrow{cTM} & T^3M \\
 & & & & & & \downarrow T(K) \\
 & & & & & & T^2M \\
 & & & & & & \downarrow K \\
 & & & & & & TM
 \end{array}$$

# New curvature definition

## Definition

For a connection  $(K, H)$  on  $M$ , its **curvature** is the map

$$T_3M \xrightarrow{\langle \langle \pi_0, \pi_1 \rangle H, \langle \pi_0, \pi_2 \rangle H \rangle T(H)cT(K)K} TM.$$

It is **flat** (curvature-free) if this is zero (that is, it equals  $\pi_0 p_0$ ).

- Again, solves all three problems, and seems to be new.

# Conclusions

- Curvature and torsion can be defined for tangent-bundle connections in a tangent category without requiring negatives.
- This may lead to new ideas in some of the examples without negatives (eg., tropical geometry, functor calculus).
- Still more work to do understanding curvature for differential bundles and more general bundles.