Equivariant Motion Planning

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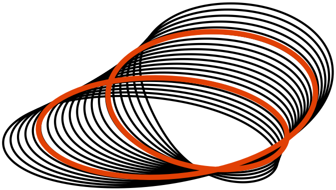

Topological Robotics

A new discipline at the intersection of topology, engineering and computer science that

1. studies pure topological problems inspired by robotics and
2. uses topological ideas and algebraic topology tools to solve problems of robotics.

Motion Planning Problem (MPP)

Robot: A mechanical system capable of moving autonomously.

Physical space: The real world X where the robot can move.

MPP: Given an initial position A and a final position B, find a path in X that moves the robot from A to B.

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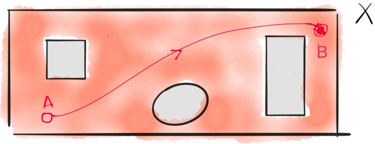
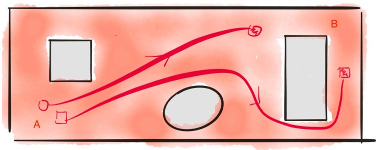


Several robots

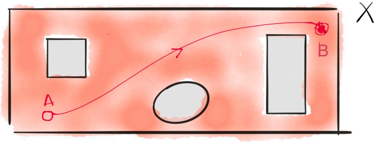


Configuration Space

A configuration is a specific state of a system; and the configuration space is the collection of all possible configurations for a given system.

Example

If a point robot moves in a physical space X, then the configuration space $C^1(X)$ is just X.

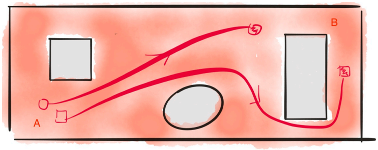


state of the system = position of the robot

Configuration Space - two robots

Example

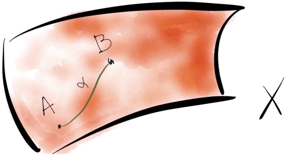
If two robots move in a physical space X, then the configuration space $C^2(X) = X \times X - \Delta$



state of the system $=$ *combined position of both robots*

Motion Planning Algorithm (MPA)

A MPA is a function that assigns to each pair of configurations A and B, a continuous motion α from A to B.

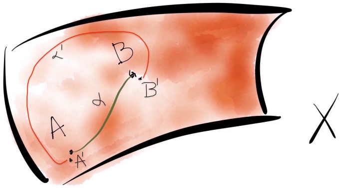


Definition

Let PX be the space of paths in X and $\mathrm{ev} : PX \to X \times X$ be the evaluation map, $\mathrm{ev}(\alpha) = (\alpha(0), \alpha(1))$. A **MPA** is a section $s : X \times X \to PX$ of ev, i.e. $\mathrm{ev} \circ s = \mathrm{id}$.

Does this section exist?

- If the space is connected, yes. Otherwise, there is no MPA.
- But even when the section exists, a fundamental question related to the stability of robot behavior is about its continuity.



Theorem (Farber)

A continuous motion planning algorithm $s: X \times X \to PX$ *exists if and only if the space* X *is contractible.*

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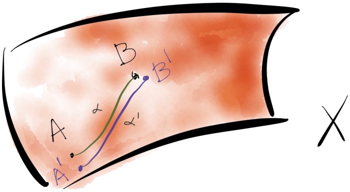

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A continuous motion planning algorithm $s : X \times X \to PX$ *exists if and only if the space* X *is contractible.*

Farber's Topological Complexity

Definition

The topological complexity $\mathbf{TC}(X)$ is the least integer k such that $X \times X$ may be covered by k open sets $\{U_1, \ldots, U_k\}$, on each of which there is a continuous section $s_i : U_i \to PX$ such that

$$\mathrm{ev} \circ s_i = i_{U_i} : U_i \hookrightarrow X \times X.$$

If no such integer exists then we set $\mathbf{TC}(X) = \infty$.

Topological complexity is a homotopy invariant.

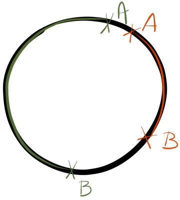
$TC(S^1) = 2$

Domains of continuity:

1. $U = \{(x, y) \in S^1 \times S^1 |\ x \text{ is not antipodal to } y\}$
2. $V = \{(x, y) \in S^1 \times S^1 |\ x \text{ is not equal to } y\}$

MPA:

1. $s_1 : U \to PX$ such that $s_1(A, B) =$shortest path between A and B.
2. $s_2 : V \to PX$ such that $s_2(A, B) =$counterclockwise path between A and B.



TC as a sectional category

Definition (1930)

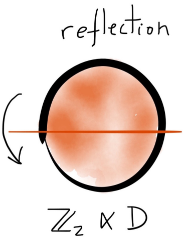
The Lusternik-Schnirelmann category of a space X, $\mathrm{cat}(X)$, is the least integer k such that X may be covered by k open sets $\{U_1, \ldots, U_k\}$, each of which is contractible in X.

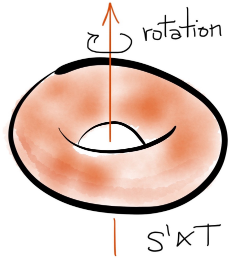
Definition (1960)

The sectional category of a fibration $p : E \to B$, $\mathrm{secat}(p)$, is the least integer k such that B may be covered by k open sets $\{U_1, \ldots, U_k\}$ on each of which there exists a map $s : U_i \to E$ such that $ps = i_{U_i} : U_i \hookrightarrow B$.

We have that $TC(X) = \mathrm{secat}(\mathrm{ev} : PX \to X \times X)$

A group G acting on the space X





Translation groupoid $G \ltimes X$ with

$$\text{objects} \quad (G \ltimes X)_0 = X$$
$$\text{arrows} \quad (G \ltimes X)_1 = G \times X$$

Equivariant map $\varphi \ltimes f \colon G \ltimes X \to K \ltimes Y$

$$f \colon X \to Y, \varphi : G \to K$$

$$f(gx) = \varphi(g) f(x)$$

All maps continuous.

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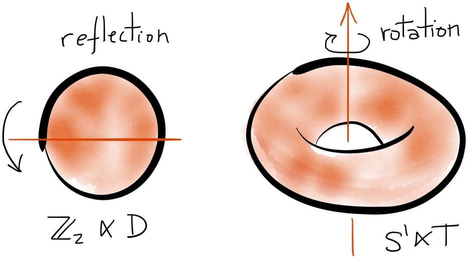

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Notions of equivalences for group actions

1. Natural equivalence:

$$G \ltimes X \underset{\psi \ltimes h}{\overset{\varphi \ltimes f}{\rightleftarrows}} K \ltimes Y$$

with $(\psi \ltimes h) \circ (\varphi \ltimes f) \cong \mathrm{id}_{G \ltimes X}$ and $(\varphi \ltimes f) \circ (\psi \ltimes h) \cong \mathrm{id}_{K \ltimes Y}$ where $\cong$ means equivalent by a natural transformation.

2. Morita equivalence:

$$G \ltimes X \xleftarrow{\psi \ltimes \sigma} J \ltimes Z \xrightarrow{\varphi \ltimes \epsilon} K \ltimes Y.$$

with $\psi \ltimes \sigma$ and $\varphi \ltimes \epsilon$ essential equivalences.

Essential Equivalence $\varphi \ltimes \epsilon : G \ltimes X \to K \ltimes Y$

1. (essentially surjective) $\phi' \circ \pi$ is an open surjection:

$$\begin{tikzcd} X \times_Y (K \times Y) \arrow[r, "\pi"] \arrow[d] & K \times Y \arrow[r, "\phi'"] \arrow[d, "p_2"] & Y \\ X \arrow[r, "\epsilon"] & Y & \end{tikzcd}$$

2. (fully faithful) the following diagram is a pullback:

$$\begin{tikzcd} G \times X \arrow[r, "\varphi \times \epsilon"] \arrow[d, "(p_2, \phi)"] & K \times Y \arrow[d, "(p_2, \phi')"] \\ X \times X \arrow[r, "\epsilon \times \epsilon"] & Y \times Y \end{tikzcd}$$

$G \times X = \{((k, y), (x, x'))|y = \epsilon(x), ky = \epsilon(x')\}.$

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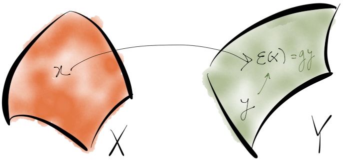

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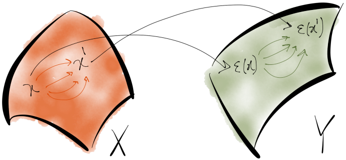

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$G \times X = \{((k, y), (x, x')) | y = \epsilon(x), ky = \epsilon(x')\}.$

An ee has to reach to all orbits and there is a bijection induced by φ: $\{g \in G | x' = gx\} \to \{k \in K | \epsilon(x') = k\epsilon(x)\}$.

Morita Equivalence $\sim$

Two actions $G \times X \to X$ and $K \times Y \to Y$ are Morita equivalent if there is a third action $J \times Z \to Z$ and two essential equivalences

$$G \ltimes X \xleftarrow{\psi \ltimes \sigma} J \ltimes Z \xrightarrow{\varphi \ltimes \epsilon} K \ltimes Y.$$

We write $G \ltimes X \sim K \ltimes Y$.

Any notion relevant to the geometric object defined by the action, should be invariant under Morita equivalence.

Examples

1. Let G be a topological group, then

$$e \ltimes X \sim G \ltimes (G \times X)$$

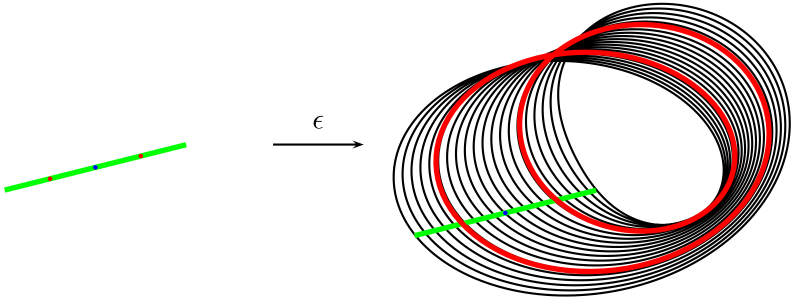
2. If H is a subgroup of G acting on X, then

$$H \ltimes X \sim G \ltimes (G \times_H X)$$

   where $[gh, x] = [g, hx]$.

Example $\mathbb{Z}_2 \ltimes I \sim_\epsilon S^1 \ltimes M$

There is an essential equivalence between the mirror action of $\mathbb{Z}_2$ on the interval $I = (-1, 1)$ and the action of S^1 on the Moebius band M.



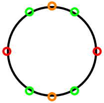
Examples

1. If G acts freely on X, then $G \ltimes X \sim e \ltimes X/G$
2. If $H \trianglelefteq G$ acts freely on X, then $G \ltimes X \sim G/H \ltimes X/H$

Example $(\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes S^1 \sim \mathbb{Z}_2 \ltimes S^1$

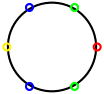
There is an essential equivalence between the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on the circle by rotation+reflection and the action of $\mathbb{Z}_2$ on S^1 by just reflection.

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{\mathsf{e}, \rho, \sigma, \rho\sigma\}$$



acting on S^1

$$\mathbb{Z}_2\times\mathbb{Z}_2 / _{<\rho>} =< \sigma >= \mathbb{Z}_2$$



acting on $S^1 / _{<\rho>} = S^1$

Pronk-Scull characterization

Any essential equivalence is a composite of maps as below:

1. (quotient map) $G \ltimes X \to G/K \ltimes X/K$
   where $K \trianglelefteq G$ and K acts freely on X.
2. (inclusion map) $K \ltimes Z \to H \ltimes (H \times_K Z)$
   where $K \leq H$ acting on Z and $H \times_K Z = H \times Z/\sim$ with $[hk, z] \sim [h, kz]$ for any $k \in K$.

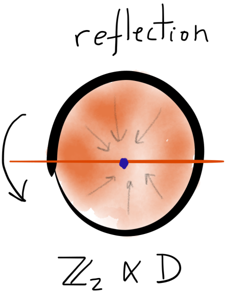
Equivariant LS-category

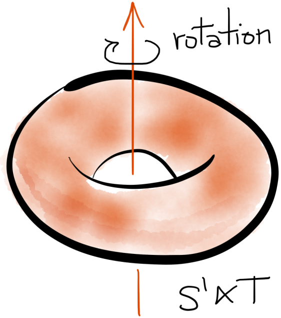
The equivariant category of a G-space X, $\mathrm{cat}_G(X)$, is the least integer k such that X may be covered by k invariant open sets $\{U_1, \ldots, U_k\}$, each of which is G-compressible into a single orbit.

That is, inclusion map $i : U \to X$ is G-homotopic to a G-map $c : U \to X$ with $c(U) \subseteq \mathrm{orb}_G(z)$ for some $z \in X$.

$$\begin{tikzcd} U \arrow[rr, "\text{i}"] \arrow[dr] & & X \\ & \mathrm{orb}_G(z) \arrow[ur] & \end{tikzcd}$$

Examples





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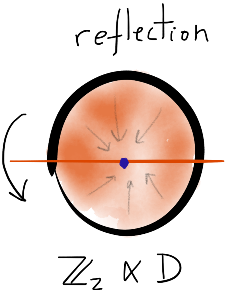

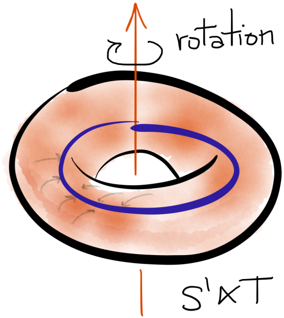

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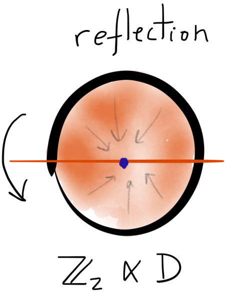

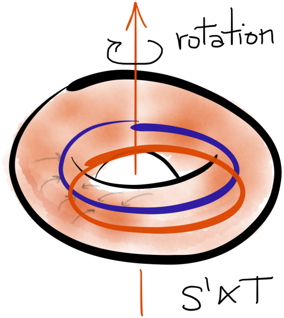

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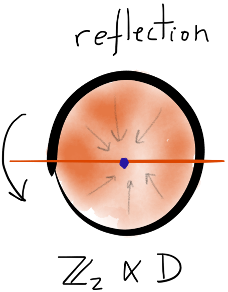

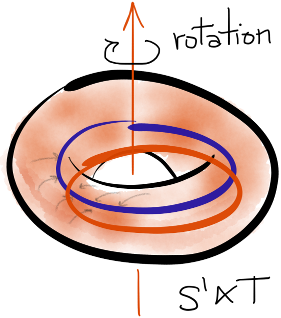

$$\mathrm{cat}_{\mathbb{Z}_2} D = 1 \qquad \mathrm{cat}_{S^1} T = 2$$

Equivariant Clapp-Puppe $\mathcal{A}$-category

Let $\mathcal{A}$ be a class of G-invariant subsets of X. The equivariant $\mathcal{A}$-category, ${}_{\mathcal{A}}cat_G(X)$, is the least integer k such that X may be covered by k G-invariant open sets $\{U_1, \ldots, U_k\}$, each G-compressible into some space $A \in \mathcal{A}$.

$$\begin{tikzcd} U \arrow[rr, "i"] \arrow[dr] & & X \\ & A \arrow[ur] & \end{tikzcd}$$

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$$\begin{tikzcd} U \arrow[rr, "i"] \arrow[dr] & & X \\ & A \arrow[ur] & \end{tikzcd}$$

In particular, ${}_{\mathcal{A}}cat_G(X) = \mathrm{cat}_G(X)$ when $\mathcal{A} =$ orbits.

The G-sectional category (Colman-Grant)

The equivariant sectional category of a G-map $p : E \to B$, $\mathrm{secat}_G(p)$, is the least integer k such that B may be covered by k invariant open sets $\{U_1, \ldots, U_k\}$ on each of which there exists a G-map $s : U_i \to E$ such that $ps \simeq_G i_{U_i} : U_i \hookrightarrow B$.

$$\begin{tikzcd} & E \arrow[d] \\ U \arrow[ur] \arrow[r, hook] & B \end{tikzcd}$$

Equivariant Motion Planning versions

Equivariant Motion Planning Problem?

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Given configurations c_0 and c_1, find a path of configurations between a and b, such that a is in the orbit of c_0 and b is in the orbit of c_1.

Equivariant Motion Planning versions

Equivariant Motion Planning Problem?

Given configurations c_0 and c_1, find a path α of configurations between c_0 and c_1, such that the path between configurations gc_0 and gc_1 is $g\alpha$.

Equivariant TC (Colman-Grant)

$$G \times PX \to PX, \qquad G \times (X \times X) \to X \times X,$$

$$g(\gamma)(t) = g(\gamma(t)), \qquad g(x, y) = (gx, gy).$$

The equivariant topological complexity of X, $\mathrm{TC}_G(X)$, is the least integer k such that $X \times X$ may be covered by k G-invariant open sets $\{U_1, \ldots, U_k\}$, on each of which there is a G-equivariant map $s_i : U_i \to X^I$ such that the diagram commutes:

$$\begin{tikzcd} & PX \arrow[d, "\mathrm{ev}"] \\ U_i \arrow[ur, "s_i"] \arrow[r] & X \times X \end{tikzcd}$$

Equivariant TC as $\mathcal{A}$-category

Theorem

For a G-space X, the following statements are equivalent:

1. $\mathrm{TC}_G(X) \leq n$.
2. $\mathrm{secat}_G(ev) \leq n$*: there exist G-invariant open sets* $U_1, \ldots, U_k$ *which cover* $X \times X$ *and G-equivariant sections* $s_i : U_i \to X^I$ *such that* $ev \circ s_i$ *is G-homotopic to* $U_i \to X \times X$.
3. ${}_{\Delta(X)}\mathrm{cat}_G(X \times X) \leq n$*: there exist G-invariant open sets* $U_1, \ldots, U_k$ *which cover* $X \times X$ *which are G-compressible into* $\Delta(X)$.

$\mathrm{TC}_G(X)$ is NOT invariant under Morita equivalence.

Invariant TC (Lubawski-Marzantowicz)

$$P'X = PX \times_{X/G} PX = \big\{(\alpha,\beta) \in PX \times PX \colon G\alpha(1) = G\beta(0)\big\}$$

$\mathrm{ev}' \colon P'X \to X \times X$ given by $\mathrm{ev}(\alpha,\beta) = \big(\alpha(0), \beta(1)\big)$ is a $(G \times G)$-fibration.

The invariant topological complexity of X, $\mathrm{TC}^G(X)$, is the least integer k such that $X \times X$ may be covered by k $(G \times G)$-invariant open sets $\{U_1, \ldots, U_k\}$, on each of which there is a $(G \times G)$-equivariant section $s_i : U_i \to P'X$ such that the diagram commutes:

$$\begin{tikzcd} & P'X \arrow[d, "\mathrm{ev}'"] \\ U_i \arrow[ur, "s_i"] \arrow[r] & X \times X \end{tikzcd}$$

Invariant TC as $\mathcal{A}$-category

Let $\Delta^{G\times G}(X)$ be the saturation of the diagonal $\Delta(X)$ with respect to the $(G \times G)$-action.

Theorem

For a G-space X the following are equivalent:

1. $\mathrm{TC}^G(X) \leq n$.
2. $\mathrm{secat}_{G\times G}(\mathrm{ev}') \leq n$*: there exist* $(G \times G)$*-invariant open sets* $U_1, \ldots, U_k$ *which cover* $X \times X$ *and* $(G \times G)$*-equivariant sections* $s_i : U_i \to PX'$ *such that* $ev \circ s_i$ *is* $(G \times G)$*-homotopic to the inclusion* $U_i \to X \times X$.
3. ${}_{\Delta^{G\times G}(X)}\mathrm{cat}_{G\times G}(X \times X) \leq n$*: there exist* $(G \times G)$*-invariant open sets* $U_1, \ldots, U_k$ *which cover* $X \times X$ *which are* $(G \times G)$*-compressible into* $\Delta^{G\times G}(X)$.

Invariance under Morita equivalence

Theorem (Angel, Colman, Grant, Oprea)

Let G be a compact Lie group, $H \leq G$ and $K \lhd G$ acting freely on X. If $\mathcal{A}$ is a class of G-invariant subsets of X, let $\mathcal{A}/K = \{A/K \mid A \in \mathcal{A}\}$ and $G \times_H \mathcal{A} = \{G \times_H A \mid A \in \mathcal{A}\}$. Then

1. ${}_{\mathcal{A}}\mathrm{cat}_G X = {}_{\mathcal{A}/K}\,\mathrm{cat}_{G/K}(X/K)$
2. ${}_{\mathcal{A}}\mathrm{cat}_H X = {}_{G\times_H\mathcal{A}}\,\mathrm{cat}_G(G \times_H X)$.

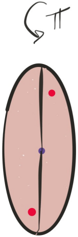
Invariance under Morita equivalence

Corollary

Let G and H be compact Lie groups. If $G \ltimes X \sim H \ltimes Y$, then

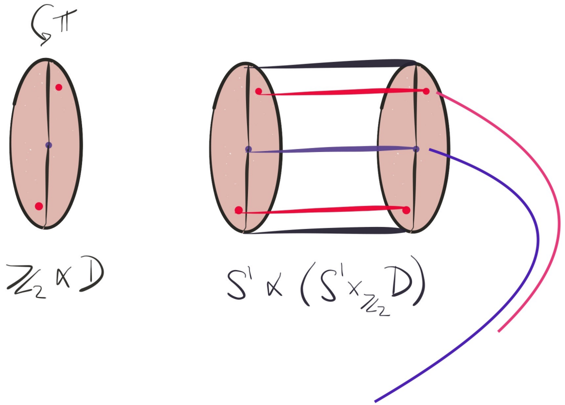
1. $\mathrm{cat}_G X = \mathrm{cat}_H Y$
2. $\mathrm{TC}^G X = \mathrm{TC}^H Y$

Seifert fibrations

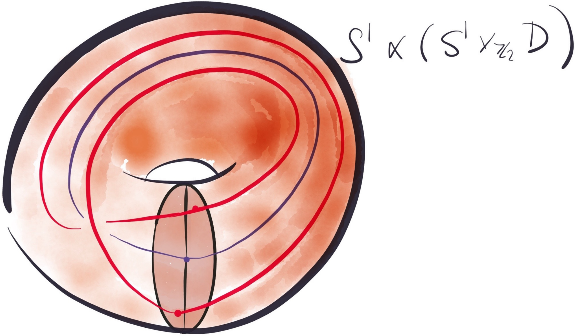


$\mathbb{Z}_2 \ltimes D$

Seifert fibrations



Seifert fibrations



$$\mathrm{cat}_{S^1}(S^1 \times_{\mathbb{Z}_2} D) = \mathrm{cat}_{\mathbb{Z}_2} D = 1$$

$$\mathrm{TC}^{S^1}(S^1 \times_{\mathbb{Z}_2} D) = \mathrm{TC}^{\mathbb{Z}_2} D = 1$$