Restriction monads and algebras.

Union College Mathematics Conference

Darien DeWolf
Dalhousie University

December 4, 2016

Monads for Partial Computation

Moggi (1991) introduces monads as abstract notions of computation.

In particular, the monad

$$T = \mathrm{Par} : \mathbf{Set} \to \mathbf{Set} : A \mapsto A \coprod \{*\}$$

can be used to re-interpret a partial function of sets $f : A \to B$ as a total function

$$f_T : A \to \mathrm{Par}(B)$$

where

$$f_T(a) = \begin{cases} f(a) & \text{if } f(a) \text{ is defined.} \\ * & \text{if } f(a) \text{ is not defined.} \end{cases}$$

Then composing partial functions can be done by composing their corresponding Kleisi arrows.

Monads for Partial Computation

Can we use monads to encode partial computation in some category other than set?

With what additional structure must we equip a monad

$$T : \mathbf{X} \to \mathbf{X}$$

to encode, in some sense, that this monad is partially defined without requiring any additional structure on the category $\mathbf{X}$ itself?

We will call such a thing a *restriction monad*.

We need to choose that sense in which a monad is partially defined.

Restriction Categories

Restriction monads will model partial computation in the same way that restriction categories (Cockett and Lack) model partial maps.

A category $\mathbf{X}$ is called a restriction category when it can be equipped with an assignment

$$(f : A \to B) \mapsto (\overline{f_A} : A \to A)$$

of all arrows f in $\mathbf{X}$ to an endomorphism $\overline{f}$ satisfying:

1. For all maps f, $f\,\overline{f_A} = f$.
2. For all maps $f : A \to B$ and $g : A \to B'$, $\overline{f_A}\,\overline{g_A} = \overline{g_A}\,\overline{f_A}$.
3. For all maps $f : A \to B$ and $g : A \to B'$, $\overline{g_A\,\overline{f_A}} = \overline{g_A}\,\overline{f_A}$.
4. For all maps $f : B \to A$ and $g : A \to B'$, $\overline{g_A}\,f = f\,\overline{(gf)_B}$.

Restriction Monads

We will define a restriction monad in a bicategory.
We need a 0-cell x with a 1-cell $T : x \to x$.

With restriction categories in mind, what additional data do we need in a restriction monad?

- We need to assign special "endomorphisms", so we need some data to hold these. We use another 1-cell $E : x \to x$ for this.
- We need to be able to pick out the "source" to anchor these endomorphisms. We use a 1-cell $D : x \to x$.

On these data, we need a "restriction operator": a 2-cell

$$\rho : D \Longrightarrow E$$

with some other 2-cells so that we can express some suitable axioms reminiscent of restriction categories.

Restriction Monads in $\mathrm{Span}(\mathbf{Set})$

Let's define these 2-cells, and look at the restriction axioms in the context of an example.

If $\mathbf{X}$ is a restriction category, consider the (ordinary) monad $R(\mathbf{X})$ in $\mathrm{Span}(\mathbf{Set})$ corresponding to its underlying category.

$$T = \begin{tikzcd} & \mathbf{X}_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \mathbf{X}_0 & & \mathbf{X}_0 \end{tikzcd}$$

$$\eta : 1_T \Rightarrow T : \mathbf{X}_0 \to \mathbf{X}_1 : A \mapsto 1_A$$

$$\begin{tikzcd} & \mathbf{X}_0 \arrow[dl, "1"'] \arrow[dr, "1"] \arrow[dd, "\eta"] & \\ \mathbf{X}_0 & & \mathbf{X}_0 \\ & \mathbf{X}_1 \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

$$\mu : T^2 \Rightarrow T : \mathbb{C} \to \mathbf{X}_1 : (f, g) \mapsto gf$$

$$\begin{tikzcd} & \mathbb{C} \arrow[dl, "s\pi_1"'] \arrow[dr, "t\pi_2"] \arrow[dd, "\mu"] & \\ \mathbf{X}_0 & & \mathbf{X}_0 \\ & \mathbf{X}_1 \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \to \mathbf{X}_0$ in $\mathrm{Span}(\mathbf{Set})$

$$D = \begin{tikzcd} & \mathbf{X}_1 \arrow[dl, "s"'] \arrow[dr, "s"] & \\ \mathbf{X}_0 & & \mathbf{X}_0 \end{tikzcd}$$

$$\Delta : T \Rightarrow TD : \mathbf{X}_1 \to \mathbb{D} : f \mapsto (f, f) \qquad \begin{tikzcd} & X_1 \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[dd, "\Delta"] & \\ \mathbf{X}_0 & & \mathbf{X}_0 \\ & \mathbb{D} \arrow[ul, "s\pi_1"] \arrow[ur, "t\pi_2"'] & \end{tikzcd}$$

$$\mathbb{D} = \{(f, g) \in \mathbf{X}_1 \times \mathbf{X}_1 : sf = sg\}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \to \mathbf{X}_0$ in $\mathrm{Span}(\mathbf{Set})$

$$E = \begin{tikzcd} & \overline{\mathbf{X}}_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \mathbf{X}_0 & & \mathbf{X}_0 \end{tikzcd}$$

where

$$\overline{\mathbf{X}}_1 = \{\overline{f} : f \in \mathbf{X}\}$$

And define

$$\rho : D \Rightarrow E : \mathbf{X}_1 \to \overline{\mathbf{X}}_1 : f \mapsto \overline{f} \qquad \begin{tikzcd} & \mathbf{X}_1 \arrow[dl, "s"'] \arrow[dr, "s"] \arrow[dd, "\rho"] & \\ \mathbf{X}_0 & & \mathbf{X}_0 \\ & \overline{\mathbf{X}}_1 \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \to \mathbf{X}_0$ in $\mathrm{Span}(\mathbf{Set})$

$\iota : E \Rightarrow T : \overline{\mathbf{X}}_1 \to \mathbf{X}_1 : \overline{f} \mapsto \overline{f}$

$$\begin{tikzcd} & \overline{\mathbf{X}}_1 \arrow[dl, "s"'] \arrow[dr, "t"] \arrow[dd, "\iota"] & \\ \mathbf{X}_0 & & \mathbf{X}_0 \\ & \mathbf{X}_1 \arrow[ul, "s"] \arrow[ur, "t"'] & \end{tikzcd}$$

$\tau : D^2 \Rightarrow D^2 : \mathbb{D} \to \mathbb{D} : (f, g) \mapsto (g, f)$

$$\begin{tikzcd} & \mathbb{D} \arrow[dl, "s\pi_1"'] \arrow[dr, "s\pi_2"] \arrow[dd, "\tau"] & \\ \mathbf{X}_0 & & \mathbf{X}_0 \\ & \mathbb{D} \arrow[ul, "s\pi_1"] \arrow[ur, "s\pi_2"'] & \end{tikzcd}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \to \mathbf{X}_0$ in $\mathrm{Span}(\mathbf{Set})$

$[\mu\,|_*\, DE] : DE \Rightarrow D : \mathbb{E}_\ell \to \mathbf{X}_1 : (\overline{f}, g) \mapsto g\overline{f}$

$$\begin{tikzcd}
& \mathbb{E}_\ell \arrow[dl, "s\pi_1"'] \arrow[dr, "s\pi_2"] \arrow[dd, "{[\mu\,|_*\, DE]}"] & \\
\mathbf{X}_0 & & \mathbf{X}_0 \\
& \mathbf{X}_1 \arrow[ul, "s"] \arrow[ur, "s"'] &
\end{tikzcd}$$

$\psi : DT \Rightarrow TD : \mathbb{C} \to \mathbb{D} : (f, g) \mapsto (gf, f)$

$$\begin{tikzcd}
& \mathbb{C} \arrow[dl, "s\pi_1"'] \arrow[dr, "t\pi_2"] \arrow[dd, "\psi"] & \\
\mathbf{X}_0 & & \mathbf{X}_0 \\
& \mathbb{D} \arrow[ul, "s\pi_1"] \arrow[ur, "s\pi_2"'] &
\end{tikzcd}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \to \mathbf{X}_0$ in $\mathrm{Span}(\mathbf{Set})$

(R.1): "$f = f\overline{f}$"

$$\begin{tikzcd} T \arrow[r, "\Delta"] \arrow[d, equal, "1_T"'] & TD \arrow[d, "T\rho"] \\ T & TE \arrow[l, "\mu . T\iota"] \end{tikzcd}$$

$$f \mapsto (f, f) \mapsto (\overline{f}, f) \mapsto f\,\overline{f}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \to \mathbf{X}_0$ in $\mathrm{Span}(\mathbf{Set})$

(R.2): "$\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$"

$$\begin{tikzcd}
D^2 \arrow[r, "\rho^2"] \arrow[dd, "\tau"'] & E^2 \arrow[dr, "\mu.\iota^2"] & \\
& & T \\
D^2 \arrow[r, "\rho^2"'] & E^2 \arrow[ur, "\mu.\iota^2"'] &
\end{tikzcd}$$

$$(f, g) \mapsto (\overline{f}, \overline{g}) \mapsto \overline{g}\,\overline{f}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \to \mathbf{X}_0$ in $\mathrm{Span}(\mathbf{Set})$

(R.3): "$\overline{g\overline{f}} = \overline{g}\,\overline{f}$"

$$\begin{tikzcd}
D^2 \arrow[r, "D\rho"] \arrow[d, "\rho^2"'] & DE \arrow[dr, "{[\mu\mid_* DE]}"] & \\
E^2 \arrow[d, "\mu.\iota^2"'] & & D \arrow[dl, "\rho"] \\
T & E \arrow[l, "\iota"] &
\end{tikzcd}$$

$$(f, g) \mapsto (\overline{f}, g) \mapsto g\overline{f} \mapsto \overline{g\overline{f}}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \to \mathbf{X}_0$ in $\mathrm{Span}(\mathbf{Set})$

(R.4): "$\overline{g} f = f \overline{gf}$"

$$\begin{tikzcd} DT \arrow[r, "\rho T"] \arrow[d, "\psi"'] & ET \arrow[r, "\mu.\iota T"] & T \\ TD \arrow[rr, "T\rho"'] & & TE \arrow[u, "\mu.T\iota"'] \end{tikzcd}$$

$$(f, g) \mapsto (gf, f) \mapsto (\overline{gf}, f) \mapsto f\,\overline{gf}$$

From $\mathrm{Span}(\mathbf{Set})$ to **rCat**

In the ordinary case, there is a bijective correspondence between small categories and monads in $\mathrm{Span}(\mathbf{Set})$.

In the case of small restriction categories and restriction monads in $\mathrm{Span}(\mathbf{Set})$, this is weakened to an adjunction[1].

This corresponds to a restriction operator being structure rather than a property; there are many ways to equip the same category with distinct restriction structures.

[1] Not in bicategories, though, need to consider these as monads in double categories à la Fiore et al.

Algebras for Restriction Monads

A right algebra (S, h) for a restriction monad consists of a 1-cell $S : x \to y$ together with 2-cells

- $h^T : ST \Rightarrow S$,
- $h^D : SD \Rightarrow S$,
- $h^E : SE \Rightarrow S$ and
- $r : D \Rightarrow SD$.

This time, $r : S \Rightarrow SD$ plays the role of restriction operator when post-composed with $S\rho$.

As well as the usual associative and unit laws, we have a host of restriction-category-flavoured conditions which are again best understood in the context of an example.

Algebras for $R(\mathbf{X})$

In $\mathrm{Span}(\mathbf{Set})$, such an algebra is a span

$$\begin{tikzcd} & A \arrow[dl, "a_1"'] \arrow[dr, "a_2"] & \\ \mathbf{X}_0 & & B \end{tikzcd}$$

with span morphisms

$$\begin{tikzcd} & \mathbf{X}_1 {}_t\!\times_{a_1} A \arrow[dl, "s\pi_1"'] \arrow[d, "h^T"] \arrow[dr, "a_2\pi_2"] & \\ \mathbf{X}_0 & A \arrow[l, "a_1"] \arrow[r, "a_2"'] & B \end{tikzcd} \qquad \begin{tikzcd} & \mathbf{X}_1 {}_s\!\times_{a_1} A \arrow[dl, "s\pi_1"'] \arrow[d, "h^D"] \arrow[dr, "a_2\pi_2"] & \\ \mathbf{X}_0 & A \arrow[l, "a_1"] \arrow[r, "a_2"'] & B \end{tikzcd}$$

$$\begin{tikzcd} & \mathbf{X}_1 {}_{t|_{\overline{\mathbf{x}}}}\!\times_{a_1} A \arrow[dl, "s|_{\overline{\mathbf{x}}}\pi_1"'] \arrow[d, "h^E"] \arrow[dr, "a_2\pi_2"] & \\ \mathbf{X}_0 & A \arrow[l, "a_1"] \arrow[r, "a_2"'] & B \end{tikzcd}$$

Algebras for $R(\mathbf{X})$

Recall that h^T is simply the action of $\mathbf{X}_1$ on A, or that algebras for T as an ordinary monad in $\mathrm{Span}(\mathbf{Set})$ is a module.

The span morphism

$$
\begin{tikzcd}
& A \arrow[dl, "a_1"'] \arrow[dr, "a_2"] \arrow[dd, "r"] & \\
\mathbf{X}_0 & & B \\
& \mathbf{X}_{1\,s}\times_{s\pi_1} A \arrow[ul, "s\pi_1"] \arrow[ur, "a_2\pi_2"'] &
\end{tikzcd}
$$

Can then be post-composed with $S\rho$:

$$a \overset{r}{\longmapsto} (f_a, ra) \overset{S\rho}{\longmapsto} (\overline{f_a}, ra)$$

So to each element of A can be associated a restriction idempotent $\overline{f_a}$. This assignment gives the module the structure of a *restriction module.*

Restriction Modules

A *restriction (left* $\mathbf{Y}$*-, right* $\mathbf{X}$*-bi)module* $\varphi : \mathbf{X} \longrightarrow\!\!\!\!\!\!\mapsto \mathbf{Y}$ is a collection

$$\{\varphi(y,x) : y \in \mathbf{Y}_0, x \in \mathbf{X}_0\}$$

of sets indexed by the objects of $\mathbf{X}$ and $\mathbf{Y}$ together with:

- for all objects $y, y' \in \mathbf{Y}$ and $x, x' \in \mathbf{X}$, a pair of action maps

$$\lambda^{\varphi}_{y',y,x} : \mathbf{Y}(y',y) \times \varphi(y,x) \longrightarrow \varphi(y',x)$$

$$\rho^{\varphi}_{y,x,x'} : \varphi(y,x) \times \mathbf{X}(x,x') \longrightarrow \varphi(y,x')$$

  We will write both $\lambda(g,\alpha)$ and $\rho(\alpha, f)$ using the dot notation $g \cdot \alpha$ and $\alpha \cdot f$.

- a map assigning each $\alpha \in \varphi(y,x)$ to some $\overline{\alpha} : x \to x$ in $\mathbf{X}$ satisfying some conditions (next slide).

Restriction Modules

Again, the conditions that this assignment of each $\alpha \in \varphi(y, x)$ to some $\overline{\alpha} : x \to x$ in $\mathbf{X}$ should not be too surprising.

1. for each $\alpha \in \varphi(y, x)$, $\overline{\alpha} = \overline{f}$ for some $f : x \to x'$ in $\mathbf{X}$;
2. for each $\alpha \in \varphi(y, x)$, $\alpha \cdot \overline{\alpha} = \alpha$;
3. for each $\alpha \in \varphi(y, x)$ and $\beta \in \varphi(y', x)$, $\overline{\alpha} \circ \overline{\beta} = \overline{\beta} \circ \overline{\alpha}$;
4. for each $\alpha \in \varphi(y, x)$ and $\beta \in \varphi(y', x)$, $\overline{\alpha \cdot \overline{\beta}} = \overline{\alpha} \circ \overline{\beta}$;
5. (a) for each $\alpha \in \varphi(y, x)$ and $f : x' \to x$ in $\mathbf{X}$, $\overline{\alpha} \circ f = f \circ \overline{\alpha \cdot f}$;
   (b) for each $\alpha \in \varphi(y, x)$ and $g : y \to y'$ in $\mathbf{Y}$, $\overline{g} \cdot \alpha = \alpha \cdot \overline{g \cdot \alpha}$.

Double Categories

We can organize these data into two double categories, related by adjunction.

	$\mathrm{rMon}(\textbf{rCat})$	$\mathrm{rMod}(\mathrm{Span}(\textbf{Set}))$
Objects	Rest. Cats.	Rest. Monads in $\mathrm{Span}(\textbf{Set})$
Vertical Arrows	Rest. Functors	Monad Morphisms
Horizontal Arrows	Rest. Modules	Algebras
Double Cells	Equivariant Maps	Equivariant Maps

$$\begin{tikzcd} \mathbf{X} \arrow[r, "M", "\shortmid" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha"] & \mathbf{X}' \arrow[d, "F'"] \\ \mathbf{Y} \arrow[r, "M'"', "\shortmid" marking] & \mathbf{Y}' \end{tikzcd}$$

$\mathrm{rMon}(\textbf{rCat})$

$$\begin{tikzcd} T \arrow[r, "A", "\shortmid" marking] \arrow[d, "F"'] \arrow[dr, phantom, "\alpha"] & T' \arrow[d, "F'"] \\ N \arrow[r, "B"', "\shortmid" marking] & N' \end{tikzcd}$$

$\mathrm{rMod}(\mathrm{Span}(\textbf{Set}))$

And these are *double restriction (bi)categories* in the sense that we can assign to each module M some $\overline{M}$ which behaves as the restriction idempotent of M.