Partial Functions and Categories of Partial Maps

Science Atlantic at Acadia University

Darien DeWolf

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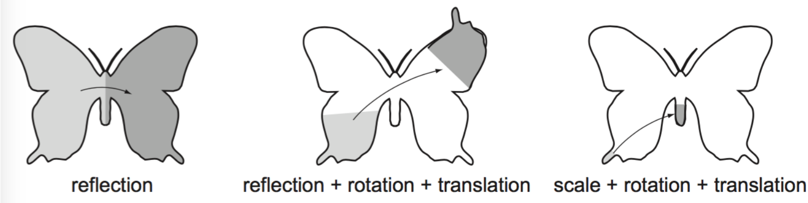
Main motivation: Partial Symmetries

Next three images all come from the following paper:

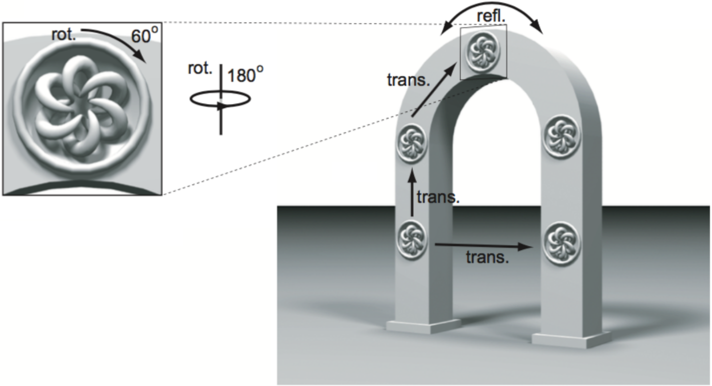
Niloy J. Mitra , Leonidas J. Guibas , Mark Pauly, *Partial and approximate symmetry detection for 3D geometry*. ACM Transactions on Graphics, **25** (3), 2006.

Click Here for Source

Nature: Butterflies



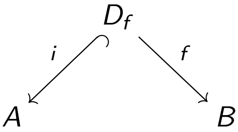
Architecture



Art

Definition: Partial Function

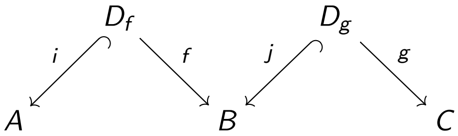
A partial function $f : A \longrightarrow\!\!\!\!\!\bullet\!\!\longrightarrow B$ (of sets) is a pair of (total, or fully defined) functions



- i is just inclusion of D_f into A.
- We think of D_f as the domain of definedness of f.
- I will call these "special spans" when I mention them again.

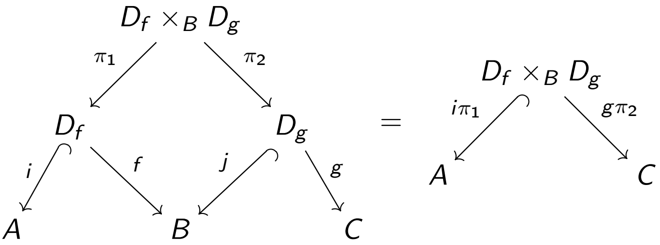
Composing Partial Functions

Compose two partial functions $A \overset{f}{-\!\bullet\!\!\rightarrow} B \overset{g}{-\!\bullet\!\!\rightarrow} C$:



Composing Partial Functions

Take the "pullback" and compose along the legs:



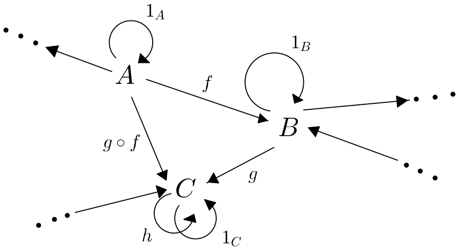
Explicitly,

$$\begin{aligned} D_f \times_B D_g &= \{(a, b) \in D_f \times D_g : b = f(a)\} \\ &= \mathrm{Im}(f) \cap D_g \end{aligned}$$

In other words: this is exactly how one is taught in pre-calculus to "find the domain" of the composite $(g \circ f)(x)$.

Definition: Category

A category contains as data: a collection of objects A, B, etc.; a collection of arrows $f : A \to B$ between objects; a composition operation defined for all suitable pairs of arrows; and identities (for composition) on each object.



Categories of Partial Maps

Par is the category of sets and partial functions:

- Objects: Sets
- Arrows: Partial functions, thought of as special spans.
- Composition: As described before.

Want to model partiality of arrows in categories whose objects are not sets. However:

- the “pullbacks” may not exist,
- a category may not even have enough objects to talk about “subobjects”.

Categories of Partial Maps

In a category, we want to phrase our definitions in terms of the arrows and their composites. This allows us to express mathematical ideas algebraically.

We can solve the object problem if we can define "domain of definedness" in terms of other partial functions (more generally, arrows), rather than relying on sets (more generally, objects).

We then will want to impose some axioms about how this domain of definedness behaves with other arrows.

Start with **Par**

In **Par**, for each partial function $f : A \longrightarrow\!\!\!\!\!\!\bullet\;\; B$, define a new partial function $\overline{f} : A \longrightarrow\!\!\!\!\!\!\bullet\;\; A$ by

$$\overline{f}(x) = \begin{cases} x, & \text{if } x \in D_f \\ \text{Not defined}, & \text{otherwise} \end{cases}$$

Intuitively, we can think of $\overline{f}$ a the domain of definedness of f. (Actually, it is exactly the identity on D_f)

This translates a set-based definition into a function-based one so that we can use it in an arbitrary category.

Restriction Categories

This assignment of $\overline{f}$ to each f satisfies the following conditions:

(i) $f\,\overline{f} = f$

(ii) $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$ if $D_f = D_g$

(iii) $\overline{g\,\overline{f}} = \overline{g}\,\overline{f}$ if $D_f = D_g$

Using these three conditions (plus one other), one can prove anything about partial functions without needing to talk about sets, elements and evaluation. This is then a suitable definition to make in an arbitrary category.

Restriction Categories

More generally, we make the following definition:

A restriction category (Cockett and Lack, 2002) is a category equipped with a restriction operator $(f : A \to B) \mapsto (\overline{f_A} : A \to A)$ satisfying

(R.1) $f\,\overline{f_A} = f$ for all f

(R.2) $\overline{f_A}\,\overline{g_A} = \overline{g_A}\,\overline{f_A}$ for all $\mathrm{dom}(f) = \mathrm{dom}(g)$

(R.3) $\overline{g_A\,\overline{f_A}} = \overline{g_A}\,\overline{f_A}$ for all $\mathrm{dom}(f) = \mathrm{dom}(g)$

(R.4) $\overline{g_A}\,f = f\,\overline{(gf)_B}$ for all $\mathrm{cod}(f) = \mathrm{dom}(g)$

Applications

Restriction categories can naturally model, for example:

- Databases with a copying operation.
- The local symmetry of spaces whose underlying "sets" may have additional structure.
- With a correct notion of "glueing" domains of definedness $\overline{f}$ and $\overline{g}$, there are natural topological data associated to restriction categories which models manifolds.

Acknowledgements

Science Atlantic
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