Restriction Monads and Category Objects

FMCS at UBC

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Double Restriction Categories

Interested in double restriction categories.

- Double cells, etc. with restriction structure in both directions. How should these behave?
- Would be helpful then to have some notion of a restriction category internal to the category of restriction categories.

Restriction Categories

A category $\mathbf{X}$ is called a restriction category when it can be equipped with an assignment

$$(f : A \to B) \mapsto (\overline{f_A} : A \to A)$$

of all arrows f in $\mathbf{X}$ to an endomorphism $\overline{f}$ satisfying:

1. For all maps f, $f\,\overline{f_A} = f$.
2. For all maps $f : A \to B$ and $g : A \to B'$, $\overline{f_A}\,\overline{g_A} = \overline{g_A}\,\overline{f_A}$.
3. For all maps $f : A \to B$ and $g : A \to B'$, $\overline{g_A\,\overline{f_A}} = \overline{g_A}\,\overline{f_A}$.
4. For all maps $f : B \to A$ and $g : A \to B'$, $\overline{g_A}\,f = f\,\overline{(gf)_B}$.

Restriction Category Objects

- Obvious data needed: object of arrows, objects, composition map, pullbacks, source and target, unit and restriction map $\rho : \mathbf{X}_1 \to \mathbf{X}_0$
- More data needed to allow us to diagrammatically express (R.1) - (R.4).
- Also, want to keep an eye out and avoid using the fact that restriction categories are internal to **Set**

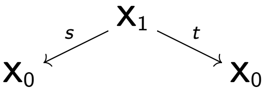
Restriction Monads: the approach

Recall that monads in **Span**(**Set**) are small categories. We will define restriction monads in a way that restriction monads in **Span**(**Set**) are small restriction categories.

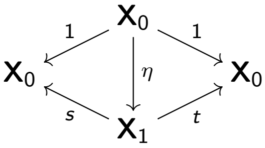
Let **X** be a restriction category and let us define a restriction monad $R(\mathbf{X})$ in **Span**(**Set**).

Restriction Monads: the construction $R(\mathbf{X})$

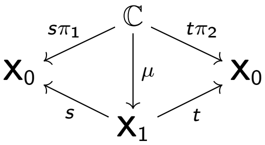
The necessary data:

$T =$ 

$\eta : 1_T \Rightarrow T : \mathbf{X}_0 \to \mathbf{X}_1 : A \mapsto 1_A$



$\mu : T^2 \Rightarrow T : \mathbb{C} \to \mathbf{X}_1 : (f, g) \mapsto gf$



Restriction Monads: the construction $R(\mathbf{X})$

Need to encode the restriction operator. A naive choice could be a morphism of spans

$$\rho : T \Rightarrow T$$

This implies that

$$sf = s(\rho f) \text{ and } tf = t(\rho f)$$

Restriction Monads: the construction $R(\mathbf{X})$

Attempting to fix this, define instead

$$\rho : D \Rightarrow T$$

where

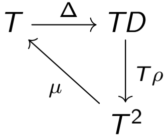
$$D = \quad \mathbf{X}_0 \xleftarrow{\;s\;} \mathbf{X}_1 \xrightarrow{\;s\;} \mathbf{X}_0$$

This implies that

$$sf = s(\rho f) \text{ and } sf = t(\rho f)$$

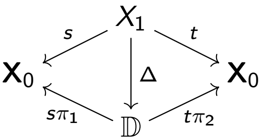
Restriction Monads: the construction $R(\mathbf{X})$

Using this new 2-cell we can express (R.1) diagrammatically by imposes that the following diagram commutes:



where

$$\Delta : T \Rightarrow TD : \mathbf{X}_1 \to \mathbb{D} : f \mapsto (f, f)$$



$$\mathbb{D} = \{(f, g) \in \mathbf{X}_1 \times \mathbf{X}_1 : sf = sg\}$$

Restriction Monads: the construction $R(\mathbf{X})$

Consider encoding (R.3): $\overline{g\overline{f}} = \overline{g}\,\overline{f}$, where $\mathrm{dom}f = \mathrm{dom}g$. There are two places to "start":

- D^2 :

$$D^2 \xrightarrow{D\rho} DT$$

$$(f, g) \longmapsto (\overline{f}, g)$$

- TD :

$$TD \xrightarrow{T\rho} T^2$$

Both ways: you get stuck.

Restriction Monads: the construction $R(\mathbf{X})$

Solution: "weaken" when we are able to compose. Idea:
We know (in **Span**(**Set**)) that the map

$$D^2 \xrightarrow{D\rho} DT$$

$$(f, g) \longmapsto (\overline{f}, g)$$

gives what should be a composable pair. Would like to keep track of these morally, yet not typely, composable pairs.

Restriction Monads: the construction $R(\mathbf{X})$

We don't need all of the composites. Turns out we can fix this by keeping track only of composites involving restriction idempotents. This can be done with new data:

$$E = \quad \mathbf{X}_0 \xleftarrow{s} \overline{\mathbf{X}}_1 \xrightarrow{t} \mathbf{X}_0$$

where

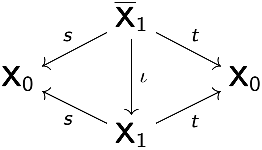
$$\overline{\mathbf{X}}_1 = \{\overline{f} : f \in \mathbf{X}\}$$

And redefine

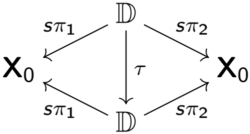
$$\rho : D \Rightarrow E$$

Restriction Monads: the construction $R(\mathbf{X})$

$$\iota : E \Rightarrow T : \overline{\mathbf{X}}_1 \to \mathbf{X}_1 : \overline{f} \mapsto \overline{f}$$

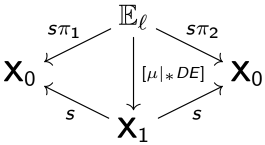


$$\tau : D^2 \Rightarrow D^2 : \mathbb{D} \to \mathbb{D} : (f, g) \mapsto (g, f)$$

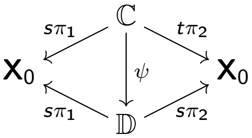


Restriction Monads: the construction $R(\mathbf{X})$

$$[\mu\,|_*\,DE] : DE \Rightarrow D : \mathbb{E}_\ell \to \mathbf{X}_1 : (\overline{f}, g) \mapsto g\overline{f}$$



$$\psi : DT \Rightarrow TD : \mathbb{C} \to \mathbb{D} : (f, g) \mapsto (gf, f)$$



Restriction Monads: the construction $R(\mathbf{X})$

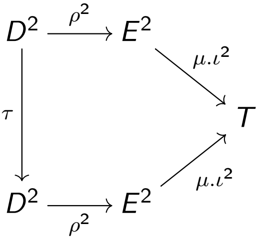
(R.1): $f = f\overline{f}$

$$\begin{array}{ccc} T & \xrightarrow{\ \Delta\ } & TD \\ {\scriptstyle 1_T}\Big\| & & \Big\downarrow{\scriptstyle T\rho} \\ T & \xleftarrow[\ \mu . T\iota\ ]{} & TE \end{array}$$

$$f \mapsto (f, f) \mapsto (\overline{f}, f) \mapsto f\,\overline{f}$$

Restriction Monads: the construction $R(\mathbf{X})$

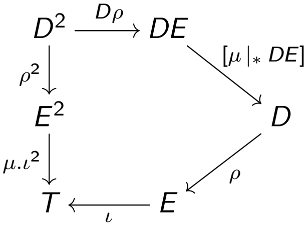
(R.2): $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$



$$(f, g) \mapsto (\overline{f}, \overline{g}) \mapsto \overline{g}\,\overline{f}$$

Restriction Monads: the construction $R(\mathbf{X})$

(R.3): $\overline{g\overline{f}} = \overline{g}\,\overline{f}$



$$(f,g) \mapsto (\overline{f},g) \mapsto g\overline{f} \mapsto \overline{g\overline{f}}$$

Restriction Monads: the construction $R(\mathbf{X})$

(R.4):

$$\begin{array}{ccccc}
DT & \xrightarrow{\rho T} & ET & \xrightarrow{\mu.\iota T} & T \\
\Big\downarrow{\scriptstyle \psi} & & & & \Big\uparrow{\scriptstyle \mu.T\iota} \\
TD & & \xrightarrow[T\rho]{\qquad\qquad\qquad} & & TE
\end{array}$$

$$(f, g) \mapsto (gf, f) \mapsto (\overline{gf}, f) \mapsto f\,\overline{gf}$$

Restriction Monads: a definition

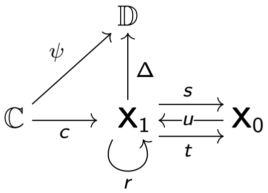
In a bicategory with involution, a restriction monad consists of a 0-cell x, 1-cells $T, D, E : x \to x$ and 2-cells

- $\eta : 1_T \Rightarrow T$,
- $\mu : T^2 \Rightarrow T$, $[\mu\,|_*\, DE] : DE \Rightarrow D$,
- $\rho : D \Rightarrow E$ (epic),
- $\iota : E \Rightarrow T$ (monic),
- $\Delta : T \Rightarrow TD$, $\tau : D^2 \Rightarrow D^2$ and
- $\psi : DT \Rightarrow TD$

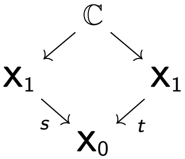
satisfying conditions (R.1) through (R.4) plus the usual monad laws plus $D^* D = DD^*$

Restriction Category Objects

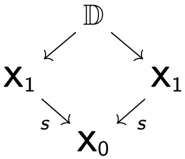
A restriction category (in **Set**) contains the following data:



where $\mathbb{C}$ and $\mathbb{D}$ are defined by the pullback squares



and



Restriction Category Objects

(i) $sr = s = tr$,

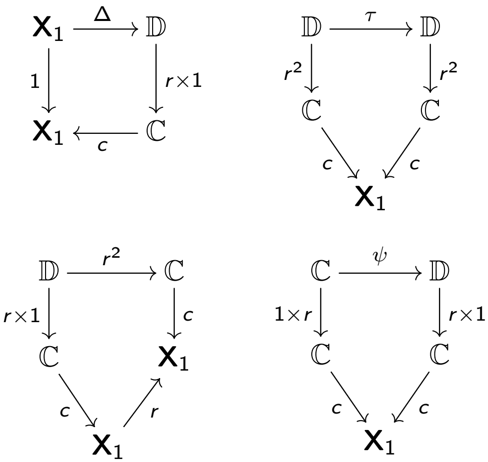
(ii) $c = \pi_1\psi$ and $\pi_1 = \pi_2\psi$,

(iii) associativity and unit laws from categories,

(iv) $\pi_1\Delta = 1 = \pi_2\Delta$,

(v) $\pi_1 = \pi_2\tau$ and $\pi_2 = \pi_1\tau$,

Restriction Category Objects



Restriction Category Objects

Definition

A double restriction category is a restriction category internal to **rCat**.

Thank you.