Restriction bicategories: two approaches

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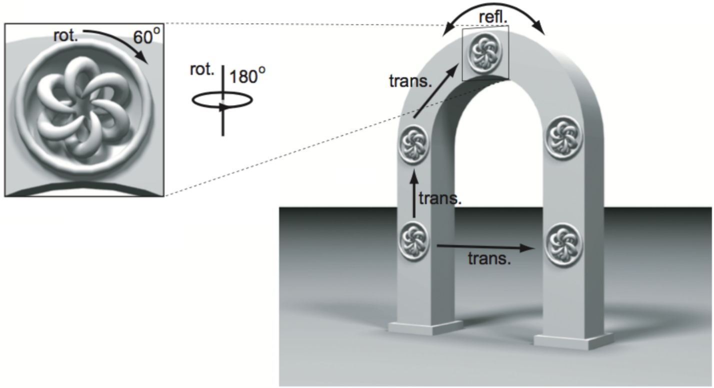
Motivating examples: partial symmetries

Consider this beautiful archway [1]:



[1] Niloy J. Mitra , Leonidas J. Guibas , Mark Pauly, *Partial and approximate symmetry detection for 3D geometry*. ACM Transactions on Graphics, **25** (3), 2006.

Motivating examples: partial symmetries

The symmetry group of this arch is $\mathbb{Z}_2$, given by horizontal reflection.

But there is clearly more going on here:

- We see a $\mathbb{Z}_5$ by moving the circular things around.
- Each of the circular things has a $\mathbb{Z}_6$ by rotation.

Inverse semigroups and inverse categories.

The solution to this problem goes back to Ehresmann, who thought it wise to model partial symmetries with with what he called the pseudogroup of transformations: the set of partial automorphisms on a set.

The point: groups aren't flexible enough to capture all of the symmetries which may be interesting to study or are instantly recognizable by humans.

The pseudogroup of partial automorphisms and its action on a set is the motivating example for studying inverse semigroups and inverse categories.

The action of an inverse semigroup on the arch captures these symmetries, but it still has a very group-like flavour.

Towards partial structures.

Rather than a total(ly defined) action of an inverse semigroup on a set, I wanted to be able to say something like “a partially defined action of a group on a set.

For example, we might want to say that $\mathbb{Z}_5$ acts partially on the garden gate, where this action is defined only on the regions containing the circular decorations.

A natural choice of structure to study, then, may be “partial modules”.

Restriction categories acting on sets.

It turns out that a convenient way to think to of partial modules is as certain modules between restriction categories.

Definition (Cockett and Lack, 2002)

A restriction structure on a category $\mathbf{X}$ is an assignment of an arrow $\overline{f} : A \to A$ to each arrow $f : A \to B$ in $\mathbf{X}$ satisfying the following four conditions:

(R.1) For all maps f, $f\,\overline{f} = f$.

(R.2) For all maps $f : A \to B$ and $g : A \to B'$, $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$.

(R.3) For all maps $f : A \to B$ and $g : A \to B'$, $\overline{g\,\overline{f}} = \overline{g}\,\overline{f}$.

(R.4) For all maps $f : B \to A$ and $g : A \to B'$, $\overline{g}\,f = f\,\overline{gf}$.

A category equipped with a restriction structure is called a *restriction category*.

Restriction bimodules.

Let **X** and **Y** be restriction categories.

A *restriction (left* **Y***-, right* **X***-bi)module* $\varphi : \mathbf{X} \longrightarrow\!\!\!\!\!\!\!\!\mid\ \ \ \mathbf{Y}$ is a (**Set**)-valued) bimodule together with a map assigning each $\alpha \in \varphi(y,x)$ to some $\overline{\alpha} : x \to x$ in **X** satisfying:

(RMod.0) for each $\alpha \in \varphi(y,x)$, $\overline{\alpha}$ is a restriction idempotent of **X**.

(RMod.1) for each $\alpha \in \varphi(y,x)$, $\alpha \cdot \overline{\alpha} = \alpha$;

(RMod.3) for each $\alpha \in \varphi(y,x)$ and $\beta \in \varphi(y',x)$, $\overline{\alpha \cdot \overline{\beta}} = \overline{\alpha} \circ \overline{\beta}$;

(RMod.4)
- (a) for each $\alpha \in \varphi(y,x)$ and $f : x' \to x$ in **X**, $\overline{\alpha} \circ f = f \circ \overline{\alpha \cdot f}$;
- (b) for each $\alpha \in \varphi(y,x)$ and $g : y \to y'$ in **Y**, $\overline{g} \cdot \alpha = \alpha \cdot \overline{g \cdot \alpha}$.

Notice that the definition of restriction bimodule implies that the collage of a restriction bimodule is a restriction category.

Bicategory of restriction bimodules.

If $\mathbf{X} \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\mid\;\;} \mathbf{Y} \overset{\psi}{\longrightarrow\!\!\!\!\!\!\mid\;\;} \mathbf{Z}$ are restriction bimodules then $\psi \otimes \varphi : \mathbf{X} \longrightarrow\!\!\!\!\!\!\mid\;\; \mathbf{Z}$ is defined by the usual coequalizer diagram

$$\coprod_{y_1, y_2 \in \mathbf{Y}_0} \psi(z, y) \times \mathbf{Y}(y_1, y_2) \times \varphi(y_2, x) \underset{1 \times \rho^{\varphi}}{\overset{\lambda^{\psi} \times 1}{\rightrightarrows}} \coprod_{y \in \mathbf{Y}_0} \psi(z, y) \times \varphi(y, x) \longrightarrow (\psi \otimes \varphi)(z, x)$$

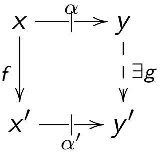
with left and right action defined by $(\beta \otimes \alpha) \cdot f = \beta \otimes (\alpha \cdot f)$ and $g \cdot (\beta \otimes \alpha) = (g \cdot \beta) \otimes \alpha$.

Proposition

If $\mathbf{X} \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\mid\;\;} \mathbf{Y} \overset{\psi}{\longrightarrow\!\!\!\!\!\!\mid\;\;} \mathbf{Z}$ *are restriction bimodules then* $\psi \otimes \varphi : \mathbf{X} \longrightarrow\!\!\!\!\!\!\mid\;\; \mathbf{Z}$ *is a restriction bimodule with* $\overline{\beta \otimes \alpha} = \overline{\overline{\beta} \cdot \alpha}$.

The restriction of a restriction bimodule.

Condition (*)
For all $f : x \to x'$ in $\mathbf{X}$, $\alpha \in \varphi(y, x)$ and $\alpha' \in \varphi(y', x')$, there exists $g : y \to y'$ such that $g \cdot \alpha = \alpha' \cdot f$. That is, the following diagram can always be completed to be commutative:



If $\varphi : \mathbf{X} \relbar\!\!\mapstochar\!\!\rightarrow \mathbf{Y}$ is a restriction bimodule satisfying condition (*), define a new module $\overline{\varphi} : \mathbf{X} \relbar\!\!\mapstochar\!\!\rightarrow \mathbf{X}$

$$\overline{\varphi}(x', x) = \{f \circ \overline{\alpha} : \alpha \in \varphi(y, x), f : x \to x'\}$$

Actions are given by composition.

Restriction bimodules of the form $\overline{\varphi}$: Four beautiful facts.

When φ and ψ have the appropriate sources and targets, we get four restriction bimodule isomorphisms:

1. $\varphi \otimes \overline{\varphi} \cong \varphi$
2. $\overline{\varphi} \otimes \overline{\psi} \cong \overline{\psi} \otimes \overline{\varphi}$
3. $\overline{\varphi \otimes \overline{\psi}} \cong \overline{\varphi} \otimes \overline{\psi}$
4. $\overline{\psi} \otimes \varphi \cong \varphi \otimes \overline{\psi \otimes \varphi}$

Takes a bit of work to show these are well defined maps (one next slide)!

But we get them, and we note that they are very much of the restriction category flavour.

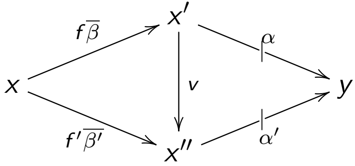
This example motivates our first approach to defining a restriction bicategory.

ρ_1 is well defined.

Define a map by

$$\rho_1 : \varphi \otimes \overline{\varphi} \to \varphi : \alpha \otimes (f\overline{\beta}) \mapsto \alpha \cdot (f\overline{\beta}).$$

This is well defined. Suppose that there is a map $v : x' \to x''$ in **X** making the following diagram commute:



Then

$$\begin{aligned} \alpha \cdot (f\overline{\beta}) &= (\alpha' \cdot v) \cdot (f\overline{\beta}) \\ &= \alpha' \cdot (v(f\overline{\beta})) \\ &= \alpha' \cdot (f'\overline{\beta'}). \end{aligned}$$

First definition of a restriction bicategory.

Let $\mathbf{X}$ be a bicategory. A restriction structure on $\mathbf{X}$ is a family of functors

$$\overline{(-)} : \mathbf{X}(A, B) \to \mathbf{X}(A, A)$$

indexed by the 0-cells of $\mathbf{X}$ together with invertible 2-cells

(i) $\rho_1 : f\overline{f} \cong f$
(ii) $\rho_2 : \overline{f}\,\overline{g} \cong \overline{g}\,\overline{f}$
(iii) $\rho_3 : \overline{g\overline{f}} \cong \overline{g}\,\overline{f}$
(iv) $\rho_4 : \overline{g}f \cong f\,\overline{gf}$

The assignment of restriction operators being functorial, in particular, implies that if $\alpha : f \cong g$ then $\overline{\alpha} : \overline{f} \cong \overline{g}$.

First definition of a restriction bicategory.

In addition to the usual pentagonal and unit coherence diagrams, the following diagrams must also commute for any isomorphism $\alpha : f \cong g$:

$$
\begin{array}{ccc}
f & \xrightarrow{\rho_1} & f\,\overline{f} \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha\overline{\alpha}} \\
g & \xrightarrow[\rho_1]{} & g\,\overline{g}
\end{array}
\qquad
\begin{array}{ccccc}
\overline{g}\,\overline{g} & \xleftarrow{\overline{\alpha}\,\overline{g}} & \overline{f}\,\overline{g} & \xrightarrow{\overline{f}\,\overline{\alpha}^{-1}} & \overline{f}\,\overline{f} \\
 & {\scriptstyle \overline{g}\,\overline{\alpha}^{-1}}\searrow & \downarrow{\scriptstyle \rho_2} & \swarrow{\scriptstyle \overline{\alpha}\,\overline{f}} & \\
 & & \overline{g}\,\overline{f} & &
\end{array}
$$

$$
\begin{array}{ccc}
\overline{g\,\overline{f}} & \xrightarrow{\rho_3} & \overline{g}\,\overline{f} \\
{\scriptstyle \overline{\alpha^{-1}\,\overline{\alpha}}}\downarrow & & \downarrow{\scriptstyle \overline{\alpha}^{-1}\overline{\alpha}} \\
\overline{f\,\overline{g}} & \xrightarrow[\rho_3]{} & \overline{g}\,\overline{f}
\end{array}
\qquad
\begin{array}{ccc}
\overline{f'}\,f & \xrightarrow{\rho_4} & f\,\overline{f'f} \\
{\scriptstyle \overline{\alpha'}\,\alpha}\downarrow & & \downarrow{\scriptstyle \alpha\,\overline{\alpha'\alpha}} \\
\overline{g'}\,g & \xrightarrow[\rho_4]{} & g\,\overline{g'g}
\end{array}
$$

.

Pros and cons of the first definition.

Pros:

- Nicely captures what we would like to call the restriction bicategory of restriction bimodules.
- Seems to be a very natural generalization of restriction categories.
- Using a natural definition of restriction enriched categories, strict restriction bicategories (2-categories) are restriction **Cat**-categories.

Cons:

- Don't know yet if these coherence conditions are enough.

Second definition of a restriction bicategory.

Cockett's idea of a restriction bicategory $\mathcal{B}$:

- Each hom-category is still equipped with a functor $\overline{(-)} : \mathcal{B}(A, B) \to \mathcal{B}(A, A)$.
- A natural transformation ι with monics $\iota_f : \overline{f} \to 1_A$.
- The conditions (R1) through (R4) are then conditions requiring that certain subobjects are equivalent. For example:
  - $f\overline{f}$ and f are equivalent as subobject of f.
  - $\overline{f}\,\overline{g}$ and $\overline{g}\,\overline{f}$ are equivalent as subobjects of 1.

Pros and cons of the second definition.

Pros:

- Since all conditions are imposed on subobjects of 1, Robin claims that coherence comes for free.
- Works very nicely for modelling restriction bimodules, since the restriction idempotents are subsets.

Cons:

- Too specific. When I look at a restriction **Cat**-category, these designated monics don't really come from anywhere and need to be additionally specified.

In progress:
A third (?) definition of a restriction bicategory.

Currently working on a middle ground, which may or may not be fruitful:

- Working mostly right now with bimodules, so don't mind too much about requiring the monics to be there.
- Rather than having the conditions be on the subobject as in Cockett's definition, would rather have conditions on how the monics interact with the restriction idempotents.
- Hopefully, the interaction between the restriction idempotents and the monics can be made so that the four restriction isomorphisms from the first definition come for free, perhaps by some universal properties.
- The four isomorphisms would then be coherent.