ON DOUBLE INVERSE SEMIGROUPS

by

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Abstract

A double semigroup is a set equipped with two associative binary operations satisfying the middle-four interchange law. A double inverse semigroup is a double semigroup in which both operations are inverse semigroup operations. It is shown by Kock [9] that all double inverse semigroups must be commutative. In this thesis, we define the notion of a double inductive groupoid which admits both a construction of double inverse semigroups given any double inductive groupoid, and vice-versa. These constructions are functorial and induce an isomorphism of categories between the category of double inductive groupoids with inductive functors and double inverse semigroups with double semigroup homomorphisms. By a further investigation of double inverse semigroups, we are able to show that the two operations of any double inverse semigroups must coincide and thus double inverse semigroups are commutative inverse semigroups.

List of Abbreviations and Symbols Used

Notation	Description
C	Denotes a category.
$\mathbf{C_0}$	Denotes the class of objects in a category **C**.
$\mathbf{C_1}$	Denotes the class of morphisms in a category **C**.
Set	Denotes the category of sets with set functions.
Cat	Denotes the category of small categories with functors.
IS	Denotes the category of inverse semigroups with semigroup homomorphisms.
IG	Denotes the category of inductive groupoids with inductive functors.
DIS	Denotes the category of double inverse semigroups with double semigroup homomorphisms.
DIG	Denotes the category of double inductive groupoids with double inductive functors.
AbPSMS	Denotes the category of presheaves of Abelian groups on meet-semilattices with presheaf morphisms.
$C_1 \times_{C_0} C_1$	If C_0 and C_1 are objects in a category with arrows $$C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0 \xrightarrow[e]{} C_1 \ ,$$ denotes the object of composable pairs of arrows and is the pullback defined by the square $$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{\pi_2} & C_1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle s} \\ C_1 & \xrightarrow[t]{} & C_0 \end{array}$$

Notation	**Description**
$\mathcal{D}$	Denotes a double category.
$\mathrm{Ver}(\mathcal{D})$	Denotes the set of vertical arrows in a double category $\mathcal{D}$.
$\mathrm{Hor}(\mathcal{D})$	Denotes the set of horizontal arrows in a double category $\mathcal{D}$.
$\mathrm{Dbl}(\mathcal{D})$	Denotes the set of double cells in a double category $\mathcal{D}$.
$s\mathrm{dom}$	Denotes the domain of s.
$s\mathrm{cod}$	Denotes the codomain of s.
$a\mathrm{hdom}$	Denotes the horizontal domain of a.
$a\mathrm{vdom}$	Denotes the vertical domain of a.
$a\mathrm{vcod}$	Denotes the vertical codomain of a.
$a\mathrm{hcod}$	Denotes the horizontal codomain of a.
$S(\odot, \circledcirc)$	Denotes a double semigroup S with operations $\odot$ and $\circledcirc$.
$\mathcal{G}$	Denotes a double inductive groupoid.
$(e_*\|a)$	Denotes the horizontal restriction of a by e.
$[e_*\|a]$	Denotes the vertical restriction of a by e.
$(a\|_*e)$	Denotes the horizontal corestriction of a by e.
$[a\|_*e]$	Denotes the vertical corestriction of a by e.
$E(S, \odot)$	Denotes the set of idempotents of a semigroup S with respect to $\odot$.

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Chapter 1

Introduction

The term "semigroup" to mean as we do – a set equipped with a single associative binary operation – is likely to have first been used in 1904, by French mathematician Séguier in [4], to describe a relaxation of group after noting several examples of group-like objects that didn't have all the properties of groups. Semigroup theory is interesting in the sense that it is deceivingly complex. A definition so innocent as a set being equipped with an associative binary operation turns out to have many non-trivial consequences due to a certain subtlety in the theory. For example, it is common knowledge there is only one group of orders 1, 2, 3 and 5 and there are only two of order 4. As a consequence of the very few conditions on semigroups, however, there exist 1, 5, 18, 126 and 1160 semigroups of order 1,2,3,4 and 5, respectively. The richness of the theory admitted some very famous and comprehensive works on the subject, including textbooks by Clifford and Preston ([2] and [3]) and Howie ([8]).

A semigroup S is said to be *inverse* if every element has a unique semigroup inverse. That is, for every $s \in S$, there is a unique $s' \in S$ such that both $ss's = s$ and $s'ss' = s'$. An element $s \in S$ is said to be *idempotent* if $ss = s$. In a group, it is known that the only idempotent is the group identity. Remarkably, though the semigroup inverse does not superficially look the same as a group inverse, it can be proven that groups are exactly inverse semigroups with only one idempotent. Though inverse semigroups and groups share this relationship, the theory of inverse semigroups inherits its complexity and richness from that of semigroups. It is exactly for this reason that Lawson's book [11], dedicated to the theory of inverse semigroups, was written.

Recall that any group G is isomorphic to some subgroup of its permutation group S_G. Another stark similarity that inverse semigroups have to groups is that there is an analogous theorem, namely the Preston-Wagner theorem, that states that every inverse semigroup X is isomorphic to some subsemigroup of its semigroup of partial isomorphisms, C_X.

1.1 Double Semigroups

In his paper, Kock [9] introduced the notion of a double semigroup. A double semigroup $(S, \circledcirc, \odot)$ is a set S equipped with two associative binary operations $\odot$ and $\circledcirc$ such that, for all $a, b, c, d \in S$, the following law, called the *middle-four interchange*, holds:

$$(a \circledcirc b) \odot (c \circledcirc d) = (a \odot c) \circledcirc (b \odot d)$$

Many examples of double semigroups exist. The most obvious example is a set equipped with the same two commutative semigroup operations. An additional example is a set equipped with left projection and right projection. Both left and right projection are associative and they satisfy the middle-four interchange. It is the case that one can combine any semigroup operation with right projection to make a double semigroup.

Kock notices that the requirement that the semigroup satisfies the middle-four interchange law implies some interesting commutativity properties of certain products. Kock proves that there exist classes of not-necessarily-commutative semigroup operations on a given set such that when two of these semigroups are compatible in the sense that they admit a double semigroup, said double semigroup is necessarily commutative.

A semigroup S is said to be cancellative if, for all $a, b, c \in S$, both $ac = bc$ implies $a = b$ and $ca = cb$ implies $a = b$. A double cancellative semigroup is a double semigroup in which both of its operations are cancellative. It is established by Kock that all double cancellative semigroups are commutative (n the sense that both of its operations are commutative).

A double inverse semigroup is a double semigroup in which both operations are inverse semigroup operations. Kock observes that all double inverse semigroups are commutative. This observation is extremely interesting, especially since it comes with (in the comments of the LaTeX source code of his paper) the musing that Kock doesn't actually have a non-trivial (i.e., both operations are not the same) example of a double inverse semigroup.

Thus starts our work in the subject. In an effort to find one, hand verification of all potential candidates up to order 3 was performed with no luck. This was done by generating a list of all commutative inverse semigroups up to order 3 using GAP [7] and then checking if any pair of them satisfy the middle-four equation. As stated above, however, the number of semigroups of a given order increases at an unmanageable rate for hand calculation.

1.2 Existence of Double Inverse Semigroups

Having failed to calculate a small example of a double inverse semigroup, our curiosity is officially piqued. Attempts to prove that non-trivial (e.g., we do not want that both operations are the same) double inverse semigroups exist fail to yield any results. Our attention then turns to a previously known construction of inverse semigroups due to Lawson [11]. Lawson successfully shows that there is an isomorphism of categories between the category of inverse semigroups with semigroup homomorphism and the category of what are called inductive groupoids – ordered groupoids with additional structure – and inductive functors. An ordered groupoid is a groupoid equipped with a partial order on its arrows that satisfies some natural conditions and also admits unique restrictions and corestrictions on arrows. An inductive groupoid is an ordered groupoid whose set of objects form a meet-semilattice.

Double categories, categories internal to **Cat**, were first introduced by Ehresmann in [6]. More specifically, he introduced double groupoids as a means of studying differential geometry. Double groupoids have also become interesting to study for other reasons. For example, Brown [1] started a web article to discuss the use of double groupoids to study higher dimensional group theory. We will be using a certain class of groupoids – double inductive groupoids – to study higher (two) dimensional inverse semigroup theory.

In an attempt to derive a similar isomorphism between the category of double inverse semigroups and double semigroup homomorphisms and the category of double inductive groupoids and double inductive functors, we first generalise the definition of an inductive groupoid to a double categorical context. This requires an inductive groupoid structure on the double cells in both the vertical and horizontal directions such that the two structures are functorial with respect to each other. That is, each of the vertical and horizontal structures interact by middle-four.

These double inductive groupoids introduce two inductive groupoid structures – one vertical and one horizontal – such that the horizontal and vertical partial orders, restrictions, corestrictions, domains, codomain and meets commute with each other, either in the sense that one distributes over the others or they satisfy middle-four. It is indeed through these double inductive groupoids that one derives the desired isomorphism between these categories. Two constructions that form the crux of this isomorphism – one for making a double inverse semigroup out of a double inductive groupoid and vice-versa – are detailed in this thesis.

Using this isomorphism, one begins to wonder exactly what structure double inductive groupoids and double inverse semigroups have. Our first remark is the implication that the two meet semilattice structures – that coming from the vertical idempotents and that coming from the horizontal idempotents – coincide on the objects of a double groupoid. It also follows from the commutativity of double inverse semigroups that any double cell in a double inductive groupoid must have its vertical domain and codomain be the same horizontal arrow, its horizontal domain and codomain the same vertical arrow and each of its four object corners be equal. This induces a natural collection of double inverse subsemigroups indexed by the objects of the double inductive groupoid. These double inverse subsemigroups are shown to be Abelian groups. It will then follow that arbitrary double inverse semigroups are presheaves of Abelian groups over meet-semilattices and thus are exactly commutative inverse semigroups.

Chapter 2

Preliminaries

2.1 Basic Category Theory

Definition 2.1.1. A *category* $\mathbf{C}$ contains the following data:

– A collection of objects, denoted $\mathrm{Obj}(\mathbf{C})$.

– For any two objects $a, b \in \mathrm{Obj}(\mathbf{C})$, a collection $\mathbf{C}(a, b)$ of arrows. We denote an arrow $f \in \mathbf{C}(a, b)$ as $f : a \to b$ or $a \xrightarrow{f} b$. These collections of arrows come together with:

  - For any three objects $a, b, c \in \mathrm{Obj}(\mathbf{C})$, an associative composition function

$$\circ : \mathbf{C}(a, b) \times \mathbf{C}(b, c) \to \mathbf{C}(a, c).$$

  - For any object $a \in \mathrm{Obj}(\mathbf{C})$, an identity arrow $1_a : a \to a$ such that, for any arrow $f : a \to b$ in $\mathbf{C}$,

$$f \circ 1_b = f = 1_a \circ f.$$

Whenever the context is such that no confusion will arise, we denote the composition $f \circ g$ using the concatenation fg. ■

Note. It is of value to note that we use postfix notation for composition of arrows. That is, we write a statement such as fg to mean "f and then g".

Example 2.1.2. We call $\mathbf{1}$ the category with one object, call it $*$, and one arrow, 1_*. ▲

Having defined a category, we recall the familiar notion of a product of categories. This is a well known construction that will be of great use later in the thesis.

Definition 2.1.3. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Define the *category product of* $\mathbf{C}$ *and* $\mathbf{D}$, denoted $\mathbf{C} \times \mathbf{D}$, to be the category containing the following data:

– The objects of $\mathbf{C} \times \mathbf{D}$ are ordered pairs (c, d), where c is an object in $\mathbf{C}$ and d is an object of $\mathbf{D}$.

– The arrows of $\mathbf{C} \times \mathbf{D}$ are ordered pairs (f, g), where f is an arrow in $\mathbf{C}$ and g is an arrow in $\mathbf{D}$.

– The composite of two arrows $(f, g) : (c, d) \to (c', d')$ and $(f', g') : (c', d') \to (c'', d'')$ is defined by $(f, g) \circ (f', g') = (f \circ f', g \circ g') : (c, c'') \to (d, d'')$. This composition is clearly associative, since it is the pairwise composition of arrows inside of two categories and thus it is coordinate-wise associative.

– For any object (c, d) in $\mathbf{C} \times \mathbf{D}$, define $1_{(c,d)} = (1_c, 1_d)$. This acts as a unit for composition since, for any arrow $(f, g) : (c, d) \to (c', d')$, we have

$$(f, g) \circ 1_{(c',d')} = (f, g) \circ (1_{c'}, 1_{d'}) = (f \circ 1_{c'}, g \circ 1_{d'}) = (f, g)$$

and

$$1_{(c,d)}(f, g) = (1_c, 1_d) \circ (f, g) = (1_c \circ f, 1_d \circ g) = (f, g).$$ ■

Definition 2.1.4. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A *functor* $F : \mathbf{C} \to \mathbf{D}$ consists of the following data:

– An object function $F : \mathrm{Obj}(\mathbf{C}) \to \mathrm{Obj}(\mathbf{D})$.

– For any $a, b \in \mathrm{Obj}(\mathbf{C})$, an arrow function $F : \mathbf{C}(a, b) \to \mathbf{D}(aF, bF)$ such that

- For any composable pair of arrows f, g in $\mathbf{C}$, $(f \circ g)F = fF \circ gF$.
- For any object $a \in \mathrm{Obj}(\mathbf{C})$, $1_aF = 1_{aF}$. ■

Note. We again draw attention to the use of postfix notation for the evaluation of F on both arrows and objects, being consistent with our use of this notation for composition.

Definition 2.1.5. Let $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ be categories and suppose we have functors $\mathbf{C} \xrightarrow{F} \mathbf{D} \xrightarrow{G} \mathbf{E}$. We naturally define the composite functor $F \circ G : \mathbf{C} \to \mathbf{E}$ as the functor whose object function is the composite $F \circ G : \text{Obj}(\mathbf{C}) \to \text{Obj}(\mathbf{E})$ and whose arrow function, for any $a, b \in \text{Obj}(\mathbf{C})$, is the composite $F \circ G : \mathbf{C}(a, b) \to \mathbf{E}(aFG, bFG)$. ■

Example 2.1.6. Let $\mathbf{C}$ be any category. We define the *identity functor on* $\mathbf{C}$, denoted $1_{\mathbf{C}}$, to be the functor such that, for any object $a \in \mathbf{C}$, $a\,1_{\mathbf{C}} = a$ and, for any arrow $f : a \to b$ in $\mathbf{C}$, $f1_{\mathbf{C}} = f$. This clearly preserves identities and composition and is thus functorial. ▲

Definition 2.1.7. Let $\mathbf{C}$ be a category. An arrow $f : a \to b$ in $\mathbf{C}$ is called an *isomorphism* whenever there exists an arrow $g : b \to a$ in $\mathbf{C}$ such that $fg = 1_a$ and $gf = 1_b$. If such a pair of arrows between two objects a and b in $\mathbf{C}$ exist, we say that a and b are *isomorphic objects*, denoted $a \cong b$. Let $\mathbf{D}$ be a category. A functor $F : \mathbf{C} \to \mathbf{D}$ is called an *isomorphism of categories* whenever there exists a functor $G : \mathbf{D} \to \mathbf{C}$ such that $FG = 1_{\mathbf{C}}$ and $GF = 1_{\mathbf{D}}$. If an isomorphism exists between the categories $\mathbf{C}$ and $\mathbf{D}$, we say that $\mathbf{C}$ and $\mathbf{D}$ are *isomorphic categories*, denoted $\mathbf{C} \cong \mathbf{D}$. ■

Example 2.1.8. Let $\mathbf{C}$ be a category. Then $\mathbf{C} \times \mathbf{1} \cong \mathbf{C} \cong \mathbf{1} \times \mathbf{C}$. We will only show that $\mathbf{C} \times \mathbf{1} \cong \mathbf{C}$. That $\mathbf{C} \cong \mathbf{1} \times \mathbf{C}$ follows analogously. We begin by defining a functor $F : \mathbf{C} \times \mathbf{1} \to \mathbf{C}$: For any object $a \in \mathbf{C}$, we define $aF = (a, *)$. For any arrow $f : a \to b$ in $\mathbf{C}$, we define $fF = (f, 1_*)$. This is functorial since, for any composable pair of arrows f and g in $\mathbf{C}$, we have that

$$(f \circ g)F = (f \circ g, 1_*) = (f \circ g, 1_* \circ 1_*) = (f, 1_*) \circ (g, 1_*) = fF \circ gF$$

and, for any object a in $\mathbf{C}$, we have that

$$1_a F = (1_a, 1_*) = 1_{(a,*)}.$$

We claim that this is an isomorphism of categories. To show this we define another functor $G : \mathbf{C} \times \mathbf{1} \to \mathbf{1}$: For any object $(a, *) \in \mathbf{C} \times \mathbf{1}$, we define $(a, *)G = a$. For any arrow $(f, 1_*) : (a, *) \to (b, *)$ in $\mathbf{C} \times \mathbf{1}$, we define $(f, 1_*)G = f$. This, too, is functorial since, for any composable pair of arrows $(f, 1_*)$ and $(g, 1_*)$ in $\mathbf{C} \times \mathbf{1}$, we have that

$$[(f, 1_*) \circ (g, 1_*)]G = (f \circ g, 1_* \circ 1_*)G = (f \circ g, 1_*)G = f \circ g$$

and, for any object $(a, *)$ in $\mathbf{C} \times \mathbf{1}$, we have that

$$(1_{(a,*)}G = (1_a, 1_*)G = 1_a = 1_{(a,*)G}.$$

Having established functoriality of G, we now check that it satisfies the composition requirements:

1. On any object $a \in \mathbf{C}$, we have $aFG = (a, *)G = a$. For any arrow $f : a \to b$ in $\mathbf{C}$, we have that $fFG = (f, 1_*)G = f$; we have established that $FG = \mathrm{id}_{\mathbf{C}}$.

2. On any object $(a, *)$ of $\mathbf{C} \times \mathbf{1}$, we have $(a, *)GF = aF = (a, *)$. For any arrow $(f, 1_*) : (a, *) \to (b, *)$ in $\mathbf{C} \times \mathbf{1}$, we have that $(f, 1_*)GF = fF = (f, 1_*)$; we have established that $GF = 1_{\mathbf{C} \times \mathbf{1}}$.

Having shown that F is indeed an isomorphism of categories, we can conclude that $\mathbf{C} \times \mathbf{1} \cong \mathbf{C}$. ▲

2.2 Semigroups

Definition 2.2.1. A *semigroup* $(S, \odot)$ is a set S equipped with an associative binary operation, $\odot$. ■

Definition 2.2.2. Two elements x and y in a semigroup S are said to be *inverse* if $x = xyx$ and $y = yxy$. A semigroup is said to be an *inverse semigroup* if every element has a unique inverse. ■

We first remind the reader of an equivalent definition, found in [11], of inverse semigroups:

Lemma 2.2.3. *A semigroup S is inverse if and only if every element has at least one inverse and all idempotents commute.* □

Notation. If S is a semigroup, we denote the set of idempotent elements in S as $E(S) = \{s \in S | s^2 = s\}$.

Definition 2.2.4. If S is a semigroup, we define the *natural order* on S with the following relation: For all $s, t \in S$, $s \leq t$ if and only if $s = et$ for some $e \in E(S)$. That is, $s \leq t$ if we can factor out an idempotent. ■

Notation. If $(S, \odot)$ is an inverse semigroup and $a \in S$, we denote the semigroup inverse of a by $a^{\odot}$.

Proposition 2.2.5. *Let $(S, \odot)$ be an inverse semigroup. Then $s^{\odot}s, ss^{\odot} \in E(S)$ for all $s \in S$.*

Proof. First, we see that $(s^{\odot}s)(s^{\odot}s) = (s^{\odot}ss^{\odot})s = s^{\odot}s$ and thus $s^{\odot}s$ is an idempotent. Similarly, $(ss^{\odot})(ss^{\odot}) = ss^{\odot}$. □

By Lemma 2.2.3, the idempotents of an inverse semigroup commute. Knowing this, we can prove the following:

Proposition 2.2.6. *Let S be an inverse semigroup. Then $(ab)^{\odot} = b^{\odot}a^{\odot}$ for all $a, b \in S$.*

Proof. We first note that $(ab)(b^{\odot}a^{\odot})(ab) = a(bb^{\odot})(a^{\odot}a)b = a(a^{\odot}a)(b^{\odot}b)b = ab$, by the commutativity of idempotents. Similarly, we find that $(b^{\odot}a^{\odot})(ab)(b^{\odot}a^{\odot}) = b^{\odot}a^{\odot}$. These two equations tell us that the inverse of ab is $b^{\odot}a^{\odot}$, or that $(ab)^{\odot} = b^{\odot}a^{\odot}$. □

Chapter 3

Double Semigroups

This chapter will introduce the notion of double semigroups. We will then explore several known results about double semigroups that will become useful in later chapters when we carry out certain constructions. In the final section of this chapter, we describe the main motivation of this thesis: the characterisation of a certain class of double semigroups.

3.1 Double Semigroups

Definition 3.1.1. A set D equipped with two associative binary operations, $\odot$ and $\circledcirc$, is called a *double semigroup* if it satisfies the middle-four interchange law. That is, for any $a, b, c, d \in D$,

$$(a \odot b) \circledcirc (c \odot d) = (a \circledcirc c) \odot (b \circledcirc d).$$ ■

Double semigroups appear at first to possibly be numerous. The restriction that the operations must satisfy the middle-four law, however, turns out to restrict the examples significantly. A first example is made:

Example 3.1.2. Any set D can be made into a double semigroup by equipping it with left and right projection. That is, $(D, \odot, \circledcirc)$ is a double semigroup where, for all $a, b \in D$, $a \odot b = a$ and $a \circledcirc b = b$. These operations are associative since, for any $a, b, c \in D$,

$$a \odot (b \odot c) = a \odot b = a = a \odot c = (a \odot b) \odot c$$

and

$$a \circledcirc (b \circledcirc c) = a \circledcirc c = c = b \circledcirc c = (a \circledcirc b) \circledcirc c.$$

It can also be seen that $\odot$ and $\circledcirc$ satisfy, for any $a, b, c, d \in D$, the interchange law:

$$(a \circledcirc b) \odot (c \circledcirc d) = b \odot d = b = a \circledcirc b = (a \odot c) \circledcirc (b \odot d). \qquad \blacktriangle$$

In a further attempt to create an example, we try to use group operations as our semigroup operations:

Example 3.1.3. Let G be a set of order 4 equipped with the group operations $\odot$ and $\circledcirc$ from $\mathbb{Z}_4 = \langle a | a^4 = 1 \rangle$ and $V_4 = \langle a, b | a^2 = b^2 = (ab)^2 = 1 \rangle$, respectively. G is *not* a double semigroup. It follows from the fact that groups are monoid structures that the units of the group structures, both denoted hereafter as 1, coincide (this follows from Theorem 3.1.4 below). Consider any non-identity element $1 \neq a \in G$ such that a is of order not equal to 2 with respect to $\mathbb{Z}_4$. Checking the interchange law, we have $(a \odot 1) \circledcirc (1 \odot a) = a \circledcirc a = 1$. However, $(a \circledcirc 1) \odot (1 \circledcirc a) = a \odot a \neq 1$; the interchange law is not satisfied. ▲

That two different group operations on the same set do not induce a double semigroup is no accident. There exists, in fact, a stronger result about what monoid structures induce double semigroups which follows directly as a corollary from the well known Eckmann-Hilton argument. The proof of the following theorem is an adaptation of the proof given by [5].

Theorem 3.1.4 (Eckmann-Hilton)**.** *Let X be a set equipped with two unital binary operations $\odot$ and $\circledcirc$. If, for all $a, b, c, d \in X$,*

$$(a \odot b) \circledcirc (c \odot d) = (a \circledcirc c) \odot (b \circledcirc d),$$

then

(i) $\odot$ *and* $\circledcirc$ *share the same unit,*

(ii) $\odot = \circledcirc$,

(iii) $\odot$ *and* $\circledcirc$ *are commutative and*

(iv) $\odot$ *and* $\circledcirc$ *are associative.*

Proof. To prove (i), we first need to recall that the unit of a unital binary operation is unique. We denote the units for $\odot$ and $\circledcirc$ as $1_\odot$ and $1_\circledcirc$, respectively. Using the interchange law, then, we see that

$$1_\odot = 1_\odot \odot 1_\odot = (1_\odot \circledcirc 1_\circledcirc) \odot (1_\circledcirc \circledcirc 1_\odot) = (1_\odot \odot 1_\circledcirc) \circledcirc (1_\circledcirc \odot 1_\odot) = 1_\circledcirc \circledcirc 1_\circledcirc = 1_\circledcirc.$$

By (i), it is unambiguous, then, that we choose to hereafter denote both units as 1.

To prove (ii), we use the interchange law to show that, for all $a, b \in X$,

$$a \odot b = (1 \circledcirc a) \odot (1 \circledcirc b) = (1 \odot 1) \circledcirc (a \circledcirc b) = 1 \circledcirc (a \circledcirc b) = a \circledcirc b.$$

By (ii), it is unambiguous, then, that we choose to hereafter denote both $a \odot b$ and $a \circledcirc b$ as simply ab, for any elements $a, b \in X$.

To prove (iii), we use the interchange law to see that, for all $a, b \in X$,

$$ab = (1a)(b1) = (1b)(a1) = ba.$$

To prove (iv), we use the interchange law to see that, for all $a, b, c \in X$,

$$a(bc) = (a1)(bc) = (ab)(1c) = (ab)c.$$

□

Corollary 3.1.5. *Let* $(D, \odot)$ *and* $(D, \circledcirc)$ *be monoids.* $(D, \odot, \circledcirc)$ *is a double semigroup if and only if* $\odot = \circledcirc$ *and both* $\odot$ *and* $\circledcirc$ *are commutative.* □

3.2 Commutativity in Double Semigroups

In his paper, Kock [9] investigates some consequences of the middle-four interchange of double semigroups. In particular, he notices that they exhibit some commutativity properties. The theorems in this section are due to Kock and the discussion and proofs very closely follow his. We begin by making some notational choices:

Notation. If $(D, \odot, \circledcirc)$ is a double semigroup, we can assign the operations each their own respective direction. That is, we can consider $\circledcirc$ as a horizontal operation and $\odot$ as a vertical operation. This is useful because it provides an easily interpreted visualisation of products in D :

- For any $a, b \in D$, we represent the product $a \circledcirc b$ horizontally as $\begin{array}{|c|c|}\hline a & b \\ \hline\end{array}$.
- For any $a, b \in D$, we represent the product $a \odot b$ vertically as $\begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array}$.

It is now noted that we can interpret products such as

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$$

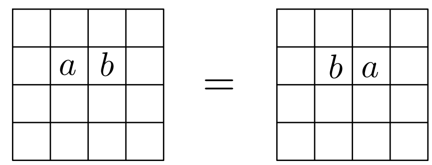
without any ambiguity because the middle four interchange law implies equality in choice of operational order (it does not matter whether we evaluate the horizontal product of the two vertical products or vice versa).

Because both $\odot$ and $\circledcirc$ are associative operations, we can rewrite large products of elements in D in several ways. If, for example, we consider the product $(a \circledcirc b) \odot (c \circledcirc d \circledcirc e)$, we can rewrite this as either $(a \circledcirc b) \odot ((c \circledcirc d) \circledcirc e)$ or $(a \circledcirc b) \odot (c \circledcirc (d \circledcirc e))$. Visually, one has that

$$\begin{array}{|c|c|c|}\hline \multicolumn{2}{|c|}{a} & b \\ \hline c & d & e \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline a & \multicolumn{2}{c|}{b} \\ \hline c & d & e \\ \hline\end{array}$$

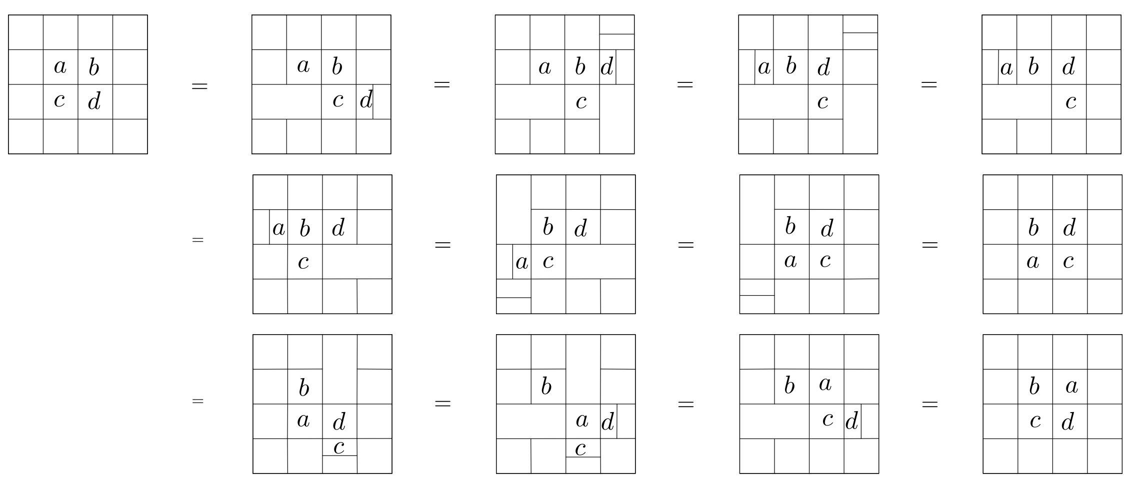
As an immediate consequence of the definition of a double semigroup, Kock establishes the following commutativity result:

Theorem 3.2.1. *For any sixteen elements $a, b, \ldots$ in any double semigroup, this equation holds:*



(The empty boxes represent fourteen nameless elements, that are the same on each side of the equation, and in the same order.)

Proof. The associativity of the two operations allows us to shift elements along any row or column independently, as we described above. It is not true in general that one may slide elements in a general array, but middle-four interchange tells us that we can multiply certain 4-tuples of elements in any order (vertical first and the horizontal, and vice-versa). In an array such as above, this condition allows us to slide the elements in either direction as long as there is some "cushion" on the outside. This cushion is in fact the outer two elements in the middle-four-shaped 4-tuple that allows us to swap the middle two. Using these facts, we can perform the following operations while fixing the border:



and the result is proved. It is of interest to note that, as long as we have that cushion, we can permute elements in any 2×2 array, as demonstrated by the first eight steps above. □

By Eckmann-Hilton, we know that double groups are not at all interesting (they are simply Abelian groups). A natural question, then, is to ask how many group properties can we add to semigroups without having the corresponding double semigroups having their operations necessarily the same (and thus reduce to Abelian groups). We consider the following class of double semigroups:

Definition 3.2.2. A semigroup S is said to be *right cancellative* if, for any $a, b, c \in S$, $ac = bc$ implies $a = b$. S is said to be *left cancellative* if, for any $a, b, c \in S$, $ca = cb$ implies $a = b$. We say that S is *cancellative* if S is both left cancellative and right cancellative. A double semigroup is said to be cancellative if both of its operations are. ■

Though not proved in his paper, Kock states the following direct result of Theorem 3.2.1 (we provide the very simple proof only to help discussion following this corollary):

Corollary 3.2.3. *A cancellative double semigroup D is commutative.*

Proof. Suppose that $a, b \in D$. Let $c \in D$ be any element of D. Then by Theorem 3.2.1,

$$\begin{array}{|c|c|c|c|}\hline c & c & c & c \\ \hline c & a & b & c \\ \hline c & c & c & c \\ \hline c & c & c & c \\ \hline \end{array} \quad = \quad \begin{array}{|c|c|c|c|}\hline c & c & c & c \\ \hline c & b & a & c \\ \hline c & c & c & c \\ \hline c & c & c & c \\ \hline \end{array}$$

and thus, by the definition of cancellative,

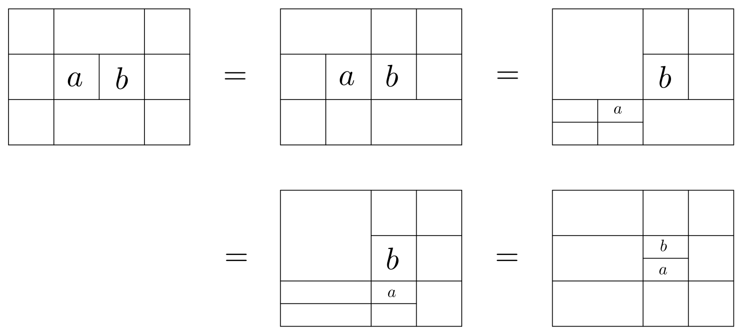
$$\begin{array}{|c|c|}\hline a & b \\ \hline \end{array} \quad = \quad \begin{array}{|c|c|}\hline b & a \\ \hline \end{array}$$

□

It is an interesting observation that in the above proof, instead of using Kock's commutativity theorem, we could have used the same method employed in his proof of the commutativity theorem to permute the middle four elements such that the vertical product, rather than the horizontal product, of a and b would be found. You could then cancel the cs in exactly the same manner as above to show that the two operations would be the same:

Proposition 3.2.4. *If* $(S, \odot, \circledcirc)$ *is a double cancellative semigroup, then* $\odot = \circledcirc$*.*

Proof (Selinger). Let $a, b \in S$ and consider the following sequence of tile slidings, where each blank square is some nameless semigroup element:



We can now apply cancellation in both directions to achieve the desired result. □

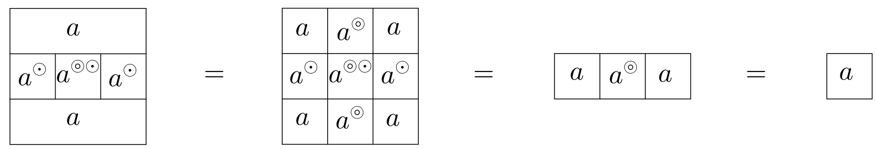
It is at first curious that a cancellative double semigroup would require two equal and commutative binary operations. The curiosity fades, however, when one recalls the well-established fact that all finite cancellative semigroups are actually groups. That is, if D is a *finite* double cancellative semigroup, we actually have an Abelian group by Theorem 3.1.4. If, however, D is *not* finite, we can still say that its two binary operations must be the same and commutative. Even without the result that double cancellative semigroups are always Abelian groups, their complete and immediate characterisation is cause for some disappointment. We will relax group properties in a different way, then, and consider the following class of double semigroups:

Definition 3.2.5. A double semigroup is called a *double inverse semigroup* if both of its operations are inverse semigroup operations. ■

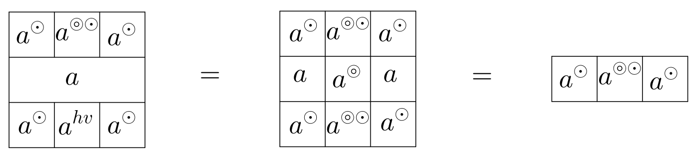
Before proving that all double inverse semigroups are commutative, we require the following lemma which is due to Kock:

Lemma 3.2.6. *Let* S *be a double inverse semigroup. Then the inverse operations of* S *commute. That is,* $a^{\odot\circledcirc} = a^{\circledcirc\odot}$ *for all* $a \in S$*.*

Proof. To prove this result, we first note that both



and



These two equations then imply that the vertical inverse of a is $a^{\odot}a^{\circledcirc\odot}a^{\odot}$, or that

$$a^{\odot} = a^{\odot}a^{\circledcirc\odot}a^{\odot}.$$

In a similar manner, we can calculate the above statements with a replaced by $a^{\circledcirc}$ to see that the vertical inverse of $a^{\circledcirc}$ is $a^{\circledcirc\odot}a^{\odot}a^{\circledcirc\odot}$, or that

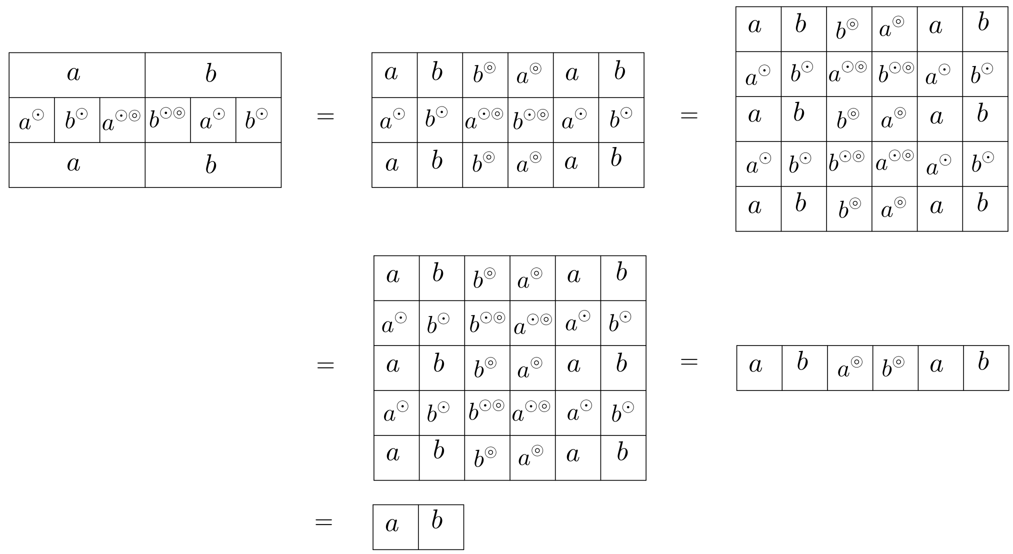
$$a^{\circledcirc\odot} = a^{\circledcirc\odot}a^{\odot}a^{\circledcirc\odot}.$$

Combining these two equations, we conclude that the horizontal inverse of $a^{\odot}$ is $a^{\circledcirc\odot}$, or $a^{\odot\circledcirc} = a^{\circledcirc\odot}$. $\square$

We can finally prove the following theorem, also due to Kock:

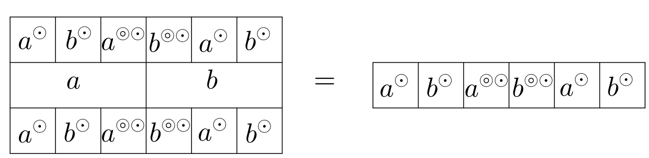
Theorem 3.2.7. *Every double inverse semigroup D is commutative.*

Proof. Let $a, b \in D$. Because inverses in either direction are unique, it suffices to show that $a^{\odot}b^{\odot} = b^{\odot}a^{\odot}$. We begin the proof by establishing the following facts:



The first step is to recognise that the horizontal inverse of ab is $(ab)^{\circledcirc} = b^{\circledcirc}a^{\circledcirc}$ and rewrite this six-fold horizontal product. We then add the bottom two rows by trading $a^{\odot}, b^{\odot}, \ldots$ with their respective conjugations with their vertical inverses. We then swap the $a^{\odot\circledcirc}$ and the $b^{\odot\circledcirc}$, which is justified by Theorem 3.2.1, since we have a 4x4 subrectangle of products. We then recognise that the columns all now collapse to a single element and we can evaluate the remaining six-fold horizontal product.

Similarly, one calculates that



These two equations together, then, imply that the vertical inverse of ab is $a^{\odot}b^{\odot}a^{\circledcirc\odot}b^{\odot\circledcirc}a^{\odot}b^{\odot}$, or that (note that $a^{\odot\circledcirc} = a^{\circledcirc\odot}$)

$$a^{\odot}b^{\odot}a^{\circledcirc\odot}b^{\circledcirc\odot}a^{\odot}b^{\odot} = a^{\odot}b^{\odot}.$$

If we replace each argument above with its horizontal inverse and do the calculations again, we find that

$$a^{\circledcirc\odot}b^{\circledcirc\odot}a^{\odot}b^{\odot}a^{\circledcirc\odot}b^{\circledcirc\odot} = a^{\circledcirc\odot}b^{\circledcirc\odot}.$$

These two equations imply that the horizontal inverse of $a^{\odot}b^{\odot}$ is $a^{\circledcirc\odot}b^{\circledcirc\odot}$. However, $(b^{\odot}a^{\odot})^{\circledcirc} = a^{\odot\circledcirc}b^{\odot\circledcirc} = a^{\circledcirc\odot}b^{\circledcirc\odot}$, too. By uniqueness of inverses, then, $a^{\odot}b^{\odot} = b^{\odot}a^{\odot}$ and we are done. □

3.3 Existence of Double Inverse Semigroups

Upon discovery that all double inverse semigroups need be commutative, one naturally asks: "Do any double inverse semigroups exist?" The answer is yes: for example, any Abelian group can be made into a double inverse semigroup by making both operations the group operation. The question, then, should rather be "Do any *interesting* (in the sense that both operations are not the same) double inverse semigroups exist?"

Using the `smallsemi` package of the GAP (Groups, Algorithms and Programming) programming language [7], all commutative inverse semigroups of order 2 and 3 were calculated and recovered. Except for group operations, it was determined by manual calculation that no non-trivial double inverse semigroups of order 2 or 3 exist; each pair failing to satisfy middle-four interchange.

In the comments of Kock's paper [9], he writes: "I should admit at this point that I don't know any significant examples of inverse double semigroups." It would appear, then, that the existence of interesting double inverse semigroups may not be so obvious.

It is this search for an example of a non-trivial double inverse semigroup that motivates the bulk of this thesis. In Chapter 6, we will introduce a known construction of inverse semigroups from what are called inductive groupoids. We will then generalise this in Chapter 7 to a construction of double inverse semigroups from double inductive groupoids in attempt to draw a correspondence between the two.

This will allow us to attempt a characterisation of double inverse semigroups. It will become apparent that the commutativity of double inverse semigroups described in Theorem 3.2.7 plays a crucial role not only in the existence of the isomorphism, but also in the characterisation of double inverse semigroups.

Chapter 4

Internal Categories

In this chapter, we will define internal categories. In the next chapter, it will be seen that categories internal to **Cat** are what are known as double categories. Also in the next chapter, this definition will be contrasted to another known definition of a double category, both definitions offering some intuition in how they work.

4.1 Internal Categories

On page 10 of [10], Mac Lane introduces a definition of a category speaking only of functions between a set of objects and a set of arrows. His definition makes no mention of the specific objects within the set of objects. This definition can be generalised in the sense that our "sets" of objects and arrows and "functions" between them could possibly be objects from some other category and morphisms between them. This generalisation will be the basis of our second definition of a double category. It is this generalisation which likely inspired the following definition, first made by Ehressman in [6]:

Definition 4.1.1. Let $\mathbf{A}$ be any complete category. A *category internal to* $\mathbf{A}$ consists of the following data:

– An object $C_0 \in \mathbf{A}_0$ of objects.

– An object $C_1 \in \mathbf{A}_0$ of morphisms.

– Source and target morphisms $s, t : C_1 \rightrightarrows C_0$.

– An identity-assigning morphism $e : C_0 \to C_1$.

– A composition morphism $c : C_1 \times_{C_0} C_1 \to C_1$.

These data are such that the following diagrams commute:

(1) Commutativity of source and target with the identity:

$$\begin{array}{ccc} C_0 & \xrightarrow{e} & C_1 \\ & {\scriptstyle \mathrm{id}_{C_0}} \searrow & \downarrow {\scriptstyle s} \\ & & C_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} C_0 & \xrightarrow{e} & C_1 \\ & {\scriptstyle \mathrm{id}_{C_0}} \searrow & \downarrow {\scriptstyle t} \\ & & C_0 \end{array}$$

(2) Commutativity of source and target with composition:

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{c} & C_1 \\ {\scriptstyle \pi_1} \downarrow & & \downarrow {\scriptstyle s} \\ C_1 & \xrightarrow[s]{} & C_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{c} & C_1 \\ {\scriptstyle \pi_2} \downarrow & & \downarrow {\scriptstyle t} \\ C_1 & \xrightarrow[t]{} & C_0 \end{array}$$

(3) Associativity of composition:

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 \times_{C_0} C_1 & \xrightarrow{c \times_{C_0} \mathrm{id}_{C_1}} & C_1 \times_{C_0} C_1 \\ {\scriptstyle \mathrm{id}_{C_1} \times_{C_0} c} \downarrow & & \downarrow {\scriptstyle c} \\ C_1 \times_{C_0} C_1 & \xrightarrow[c]{} & C_1 \end{array}$$

(4) Commutativity of left and right identity for composition:

$$\begin{array}{ccccc} C_0 \times_{C_0} C_1 & \xrightarrow{e \times_{C_0} \mathrm{id}_{C_1}} & C_1 \times_{C_0} C_1 & \xleftarrow{\mathrm{id}_{C_1} \times_{C_0} e} & C_1 \times_{C_0} C_0 \\ & {\scriptstyle \pi_2} \searrow & \downarrow {\scriptstyle c} & \swarrow {\scriptstyle \pi_1} & \\ & & C_1 & & \end{array}$$

In the preceding diagrams, pullback $C_1 \times_{C_0} C_1$ is defined via the square

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{\pi_2} & C_1 \\ {\scriptstyle \pi_1} \downarrow & & \downarrow {\scriptstyle s} \\ C_1 & \xrightarrow[t]{} & C_0 \end{array}$$

We now make two remarks: First, this pullback *always* exists in **A**, since **A** was chosen to be complete. Second, the commutativity of this diagram implies that $\pi_1 t = \pi_2 s$, or that the source of the second coordinate must be the target of the first. This matching of domain and codomain for composition is exactly what we want and thus this pullback square is the correct notion of domain for our composition morphism. ■

A good example of an internal category is a double category, which will be defined in Chapter 5. We can, however, provide a basic example of an internal category: the familiar notion of a small category.

Proposition 4.1.2. *Categories internal to* **Set** *are small categories.*

Proof. First, suppose that we are given a small category **C**. Let C_0 be the set of objects in **C** and let C_1 be the set of all arrows in **C**. Obviously, C_0 and C_1 are objects in **Set**. We now must define our required arrows (set functions) s, t, e and c :

– Define functions $s, t : C_1 \to C_0$ by $fs = f\text{dom}$ and $ft = f\text{cod}$ for all $f \in C_1$.

– Define a function $e : C_0 \to C_1$ by $xe = 1_x$ for all $x \in C_0$.

– Define a function $c : C_1 \times_{C_0} C_1 \to C_1$ by $(f, g)c = f \circ g$ for all $(f, g) \in C_1 \times_{C_0} C_1$. Note that $C_1 \times_{C_0} C_1$ is just a subset of the Cartesian product (since we are in **Set**) $C_1 \times C_1$ such that for any $(f, g) \in C_1 \times_{C_0} C_1$, we have $ft = gs$, which is exactly what we require for composition in **C**. That is, this function is indeed well defined everywhere.

We now check that the required diagrams commute:

(1) Commutativity of source and target with the identity: Take any $x \in C_0$. Then $xes = 1_x s = x$ and $x\text{id}_{C_0} = x$ and the left triangle commutes. Similarly, the right triangle commutes.

(2) Commutativity of source and target with composition: Take any $(f, g) \in C_1 \times_{C_0} C_1$. Then $(f, g)cs = (f \circ g)s = fs$ and $(f, g)\pi_1 s = fs$ and the left square commutes. Similarly, the right square commutes.

(3) Associativity of composition: Associativity of composition is directly inherited from the associativity of composition in **C**.

(4) Commutativity of left and right identity for composition: Take any $(x, f) \in C_0 \times_{C_0} C_1$. Then $fs = x$ and both $(x, f)(e \times_{C_0} \mathrm{id}_{C_1})c = (1_x, f) = 1_x \circ f = f$ and $(x, f)\pi_2 = f$. The left triangle is thus commutative. Similarly, the right triangle commutes.

Having the required functions defined that satisfy the required diagrams, **C** is a category internal to **Set**.

Conversely, suppose we are given a category **A** internal to **Set** consisting of a set of objects C_0 and a set of arrows C_1 together with the functions s, t, e and c satisfying the required diagrams. We now define a category **C** with the following data:

– A set of objects $\mathbf{C_0} = C_0$.

– For any two objects $x, y \in \mathbf{C}_0$, a set of arrows $\mathbf{C}(x, y) \subseteq C_1$. We say $f \in \mathbf{C}(x, y)$ if $fs = x$ and $ft = y$. It is clear, then, that $\mathbf{C}(x, y) \times \mathbf{C}(y, z)$ is a subset of $C_1 \times_{C_0} C_1$ and we can thus define, for any $x, y, z \in \mathbf{C}_0$, a composition function

$$\circ : \mathbf{C}(x, y) \times \mathbf{C}(y, z) \to \mathbf{C}(x, z)$$

by $f \circ g = (f, g)c$. This composition is associative by associativity of c in **A**. For any object $x \in \mathbf{C}_0$, we define an identity arrow $1_x = xe$. By the commutativity of e with c, we know that, for any $f : x \to y$,

$$f \circ 1_y = f = 1_x \circ f.$$

Having defined all required data and having checked all required conditions, we conclude that **C** is a small category. □

Chapter 5

Double Categories

In this chapter, we will define double categories in two ways: an object-arrow definition (as in the normal categorical way) and an arrows-only approach (in an internal categorical way). We will then show that these two definitions are equivalent, thus drawing insight from both descriptions. In the final section of this chapter, we attempt to view double semigroups in a double categorical perspective using a known construction for semigroups from double groupoids.

5.1 Double Categories

Definition 5.1.1. A *double category* $\mathcal{D}$ consists of the following data:

– A collection $\mathrm{Obj}(D)$ of objects.

– For any two objects $A, B \in \mathcal{D}$, a collection $\mathrm{Ver}(\mathcal{D})(A, B)$ of vertical arrows. We denote a vertical arrow $f \in \mathrm{Ver}(\mathcal{D})(A, B)$ as $f : A \overset{\bullet}{\longrightarrow} B$ or $A \overset{f}{\overset{\bullet}{\longrightarrow}} B$. These collections of vertical arrows come together with:

  - For any objects $A, B, C \in \mathcal{D}$, an associative vertical composition function

$$\bullet : \mathrm{Ver}(\mathcal{D})(A, B) \times \mathrm{Ver}(\mathcal{D})(B, C) \to \mathrm{Ver}(\mathcal{D})(A, C).$$

  - For any object $A \in \mathcal{D}$, a vertical identity arrow $1_A : A \overset{\bullet}{\longrightarrow} A$ such that, for any arrow $f : A \overset{\bullet}{\longrightarrow} B$ in $\mathrm{Ver}(\mathcal{D})$,

$$f \bullet 1_B = f = 1_A \bullet f.$$

– For any two objects $A, B \in \mathcal{D}$, a collection $\mathrm{Hor}(\mathcal{D})(A, B)$ of horizontal arrows. We denote these horizontal arrows in the usual way. These horizontal arrows come together with:

- For any objects $A, B, C \in \mathcal{D}$, an associative horizontal composition function

$$\circ : \mathrm{Hor}(\mathcal{D})(A, B) \times \mathrm{Hor}(\mathcal{D})(B, C) \to \mathrm{Hor}(\mathcal{D})(A, C).$$

- For any object $A \in \mathcal{D}$, a horizontal identity arrow $\mathrm{id}_A : A \to A$ such that, for any arrow $f : A \to B$ in $\mathrm{Hor}(\mathcal{D})$,

$$f \circ \mathrm{id}_B = f = \mathrm{id}_A \circ f.$$

– A collection $\mathrm{Dbl}(\mathcal{D})$ of double cells. A double cell α has the following form:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow{\scriptstyle v} \\ C & \xrightarrow[g]{} & D \end{array}$$

where

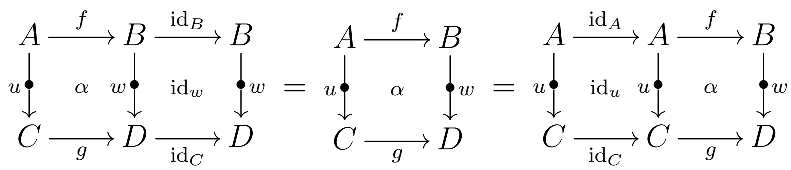
- A, B, C and D are objects of $\mathcal{D}$.
- u and v, vertical arrows, are the horizontal domain and codomain, respectively, of α. We denote these as $u = \alpha\mathrm{hdom}$ and $v = \alpha\mathrm{hcod}$.
- f and g, horizontal arrows, are the vertical domain and codomain, respectively, of α. We denote these as $f = \alpha\mathrm{vdom}$ and $g = \alpha\mathrm{vcod}$.

These doubles cells must come together with:

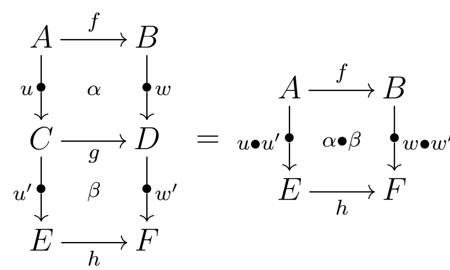
- An associative horizontal composition defined by, for $\alpha, \beta \in \mathrm{Dbl}(\mathcal{D})$,

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & C \\ {\scriptstyle u}\big\downarrow\!\bullet & \alpha & {\scriptstyle v}\big\downarrow\!\bullet & \beta & \bullet\!\big\downarrow{\scriptstyle w} \\ D & \xrightarrow[g]{} & E & \xrightarrow[g']{} & F \end{array} = \begin{array}{ccc} A & \xrightarrow{f \circ f'} & C \\ {\scriptstyle u}\big\downarrow\!\bullet & \alpha \circ \beta & \bullet\!\big\downarrow{\scriptstyle w} \\ D & \xrightarrow[g \circ g']{} & F \end{array}$$

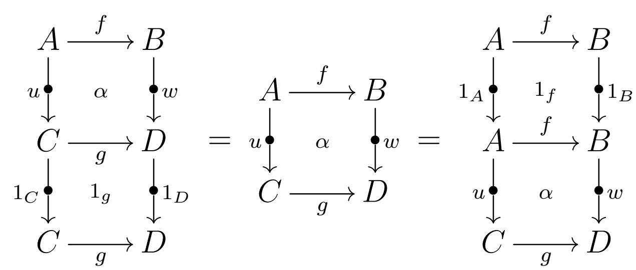
together with, for any $\alpha \in \mathrm{Dbl}(\mathcal{D})$, horizontal identity double cells such that



- An associative vertical composition defined by, for $\alpha, \beta \in \mathrm{Dbl}(\mathcal{D})$,



together with, for any $\alpha \in \mathrm{Dbl}(\mathcal{D})$, vertical identity double cells such that



- Horizontal and vertical composition of double cells must satisfy the middle-four interchange law. That is, for any $\alpha, \beta, \gamma, \delta \in \mathrm{Dbl}(\mathcal{D})$,

$$(\alpha \bullet \beta) \circ (\gamma \bullet \delta) = (\alpha \circ \gamma) \bullet (\beta \circ \delta).$$

If it is the case that $\mathrm{Ver}(\mathcal{D}) = \mathrm{Hor}(\mathcal{D})$ and the horizontal and vertical compositions are the same operation, then we can omit the dots on vertical arrows and use concatenation to denote this composition without any ambiguity. ■

Example 5.1.2. Let $\mathbf{C}$ be a category. Define $Q_4(\mathbf{C})$ to be the double category with objects $Q_4(\mathbf{C})_0 = \mathbf{C}_0$, horizontal arrows $\mathrm{Hor}(Q_4(\mathbf{C})) = \mathbf{C}_1$, vertical arrows

$\mathrm{Ver}(Q_4(\mathbf{C})) = \mathbf{C}_1$ and double cells squares

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array}$$

such that $fw = ug$.

Note that the lack of dots on the vertical arrows in the above is unambiguous, since the vertical and horizontal arrows in $Q_4(\mathbf{C})$ are the same.

We define the horizontal composite of two double cells (commutative squares) as follows:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} \\ D & \xrightarrow[h]{} & E & \xrightarrow[i]{} & F \end{array} \;=\; \begin{array}{ccc} A & \xrightarrow{f\circ g} & C \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle w} \\ D & \xrightarrow[h\circ i]{} & F \end{array}$$

We define a similar vertical composition. That middle-four is satisfied is obvious since both compositions of

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle u}\downarrow & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle w} \\ D & \xrightarrow{h} & E & \xrightarrow{i} & F \\ {\scriptstyle x}\downarrow & & {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle z} \\ G & \xrightarrow[j]{} & H & \xrightarrow[k]{} & I \end{array}$$

result in the double cell

$$\begin{array}{ccc} A & \xrightarrow{fg} & C \\ {\scriptstyle ux}\downarrow & & \downarrow{\scriptstyle wz} \\ G & \xrightarrow[jk]{} & I \end{array}$$

▲

Example 5.1.3. Let $\mathbf{C}$ be a category. Define $\mathbb{H}(\mathbf{C})$ to be the double category with objects $\mathbb{H}(\mathbf{C})_0 = \mathbf{C}_0$, horizontal arrows $\mathrm{Hor}(\mathbb{H}(\mathbf{C})) = \mathbf{C}_1$, vertical arrows $\mathrm{Ver}(\mathbb{H}(\mathbf{C})) = \{1_a : a \to a | a \in \mathbf{C}_0\}$, the identity arrows from $\mathbf{C}$, and double cells the vertical identities

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle 1_A}\big\downarrow & 1_f & \big\downarrow{\scriptstyle 1_B} \\ A & \xrightarrow[f]{} & B \end{array}$$

We define the horizontal composition of double cells (vertical identities) as follows:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle 1_A}\big\downarrow & 1_f & {\scriptstyle 1_B}\big\downarrow & 1_g & \big\downarrow{\scriptstyle 1_C} \\ A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{fg} & C \\ {\scriptstyle 1_A}\big\downarrow & 1_f \circ 1_g & \big\downarrow{\scriptstyle 1_C} \\ A & \xrightarrow[fg]{} & C \end{array}$$

We define the vertical composition analogously, which implies that the vertical structure is trivial (since it is induced by vertical composition of vertical identities). We can also see that horizontal composition is well defined: the composition of two vertical identities (double cells) gives us another vertical identity. Middle-four is trivially satisfied since

$$(1_f \circ 1_g) \bullet (1_f \circ 1_g) = 1_f \circ 1_g = (1_f \bullet 1_f) \circ (1_g \bullet 1_g). \qquad \blacktriangle$$

The preceding example is indeed not very interesting in terms of being a double category; the triviality of the vertical composition actually means that is just a (single) category. This example is useful, however, since our choice of double cells shows us that a given double category $\mathcal{D}$ admits two natural categories:

(i) A category whose objects are the horizontal arrows of $\mathcal{D}$, whose arrows are the double cells of $\mathcal{D}$ and whose composition is the vertical composition of $\mathcal{D}$. This makes sense since the vertical domains and codomains of double cells in $\mathcal{D}$ are horizontal arrows.

(ii) Similarly, there is a category whose objects are the vertical arrows of $\mathcal{D}$, whose arrows are the double cells of $\mathcal{D}$ and whose composition is the horizontal composition of $\mathcal{D}$.

The usefulness of these constructions will become extremely apparent when we are working with double inductive groupoids in Chapter 7.

Example 5.1.4. Let $\mathbf{C}$ be a 2-category. Define $Q_5(\mathbf{C})$ to be the double category with objects $Q_5(\mathbf{C})_0 = \mathbf{C}_0$, horizontal arrows $\mathrm{Hor}(Q_5(\mathbf{C}))$ the 1-cells of $\mathbf{C}$, vertical arrows $\mathrm{Ver}(Q_5(\mathbf{C}))$ the 1-cells of $\mathbf{C}$ and double cells be the 2-cells $\alpha : fw \Rightarrow ug$:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & \alpha & \downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \end{array}$$

We check that horizontal composition of 2-cells in the usual sense works in the double categorical sense. That vertical composition also works follows analogously and identities for both are obvious. If $\alpha, \beta \in \mathrm{Dbl}(Q_5(\mathbf{C})$, then the composition

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & C \\ {\scriptstyle u}\downarrow & \alpha & {\scriptstyle v}\downarrow & \beta & \downarrow{\scriptstyle w} \\ D & \xrightarrow[g]{} & E & \xrightarrow[g']{} & F \end{array} = \begin{array}{ccc} A & \xrightarrow{f\circ f'} & C \\ {\scriptstyle u}\downarrow & \alpha\circ\beta & \downarrow{\scriptstyle w} \\ B & \xrightarrow[g\circ g']{} & F \end{array}$$

is indeed well defined since there exists a 2-cell (we use juxtaposition notation for composition, since both the vertical and horizontal arrows are the same)

$$\alpha \circ \beta : ff'w \underset{f\beta}{\Longrightarrow} fvg' \underset{\alpha g'}{\Longrightarrow} ugg'$$ ▲

We now check that middle-four is satisfied. Any 4-tuple of composable cells

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle u}\downarrow & \alpha & {\scriptstyle u}\downarrow & \beta & \downarrow{\scriptstyle w} \\ D & \xrightarrow{h} & E & \xrightarrow{i} & F \\ {\scriptstyle x}\downarrow & \gamma & {\scriptstyle y}\downarrow & \delta & \downarrow{\scriptstyle z} \\ G & \xrightarrow[j]{} & H & \xrightarrow[k]{} & I \end{array}$$

is known as a pasting diagram in a 2-category. It is well known that this pasting is unique in the sense that any path one takes in evaluating this product resolves to the same cell. In particular, we could take the vertical composite of the horizontal composites or vice-versa and get equality of the two, implying that middle-four is satisfied.

5.2 Double Functors

In a later Chapter 7, we will need to consider functors between certain double categories. These functors are natural extensions of functors between categories. Instead of having only one arrows function, however, we will require three: one for each of the sets of vertical arrows, horizontal arrows and double cells. Also, we will require that the arrow function on double cells preserves identities and composition in both directions, of course. Formally, we make the following definition:

Definition 5.2.1. Let

$$\mathcal{C} = (\mathrm{Obj}(C), \mathrm{Ver}(\mathcal{C}), \mathrm{Hor}(\mathcal{C}), \mathrm{Dbl}(\mathcal{C})) \text{ and } \mathcal{D} = (\mathrm{Obj}(D), \mathrm{Ver}(\mathcal{D}), \mathrm{Hor}(\mathcal{D}), \mathrm{Dbl}(\mathcal{D}))$$

be double categories. A *double functor*, often called simply a functor, $F : \mathcal{C} \to \mathcal{D}$ contains the following data:

– An object function $F : \mathrm{Obj}(C) \to \mathrm{Obj}(D)$.

– For any $a, b \in \mathrm{Obj}(C)$, a vertical arrow function $F : \mathrm{Ver}(\mathcal{C})(a, b) \to \mathrm{Ver}(\mathcal{D})(aF, bF)$ such that

- For any composable pair of arrows $f, g \in \mathrm{Ver}(\mathcal{C})$, $(f \bullet g)F = fF \bullet gF$.
- For any object $a \in \mathrm{Obj}(C)$, $1_aF = 1_{aF}$.

– For any $a, b \in \mathrm{Obj}(C)$, a horizontal arrow function $F : \mathrm{Hor}(\mathcal{C})(a, b) \to \mathrm{Hor}(\mathcal{D})(aF, bF)$ such that

- For any composable pair of arrows $f, g \in \mathrm{Hor}(\mathcal{C})$, $(f \circ g)F = fF \circ gF$.
- For any object $a \in \mathrm{Obj}(C)$, $\mathrm{id}_aF = \mathrm{id}_{aF}$.

– For any pair (f, g) of horizontal arrows and pair (u, v) of vertical arrows, a double cell function

$$F : \{\alpha \in \mathrm{Dbl}(\mathcal{D}) | \alpha \mathrm{vdom} = f, \alpha \mathrm{vcod} = g, \alpha \mathrm{hdom} = u, \alpha \mathrm{hcod} = v\}$$
$$\to \{\alpha \in \mathrm{Dbl}(\mathcal{D}) | \alpha \mathrm{vdom} = fF, \alpha \mathrm{vcod} = gF, \alpha \mathrm{hdom} = uF, \alpha \mathrm{hcod} = vF\}$$

such that

- For any horizontally composable pair of arrows $\alpha, \beta \in \mathrm{Dbl}(\mathcal{C})$, $(\alpha \circ \beta)F = \alpha F \circ \beta F$.
- For any vertically composable pair of arrows $\alpha, \beta \in \mathrm{Dbl}(\mathcal{C})$, $(\alpha \bullet \beta)F = \alpha F \bullet \beta F$.
- For any horizontal arrow $f \in \mathrm{Hor}(\mathcal{C})$, $1_f F = 1_{fF}$.
- For any vertical arrow $u \in \mathrm{Ver}(\mathcal{C})$, $\mathrm{id}_u F = \mathrm{id}_{uF}$. ■

5.3 Categories Internal to Cat

As has been previously stated, there are two ways to consider the notion of double categories: there is the object-arrow-double cell definition from the previous section or there is the internal categorical definition, which will be presented in the following proposition. Both ways of viewing double categories offer their benefits and insights; the object-arrow-double cell definition gives us a more concrete, hands on and visual way of manipulating double categories, while the internal definition gives us a more streamlined and standard approach to constructing new types of double categories by considering them as categories internal to some category of interest. The proof of the following proposition will give insight and intuition into some of the kinds of conditions that need be satisfied by internal structures, which will help us in Chapter 7 when we are trying to define double inductive groupoids.

Proposition 5.3.1. *Categories internal to* **Cat** *are double categories.*

Proof. First, suppose that we are given a double category $\mathcal{D}$ with its accompanying data $\mathrm{Obj}(D), \mathrm{Ver}(\mathcal{D}), \mathrm{Hor}(\mathcal{D})$ and $\mathrm{Dbl}(\mathcal{D})$. We define a category of objects D_0 as a category with objects the elements of $\mathrm{Obj}(D)$ and arrows the elements of $\mathrm{Ver}(\mathcal{D})$. We define a category of arrows D_1 as a category with objects the elements of $\mathrm{Hor}(\mathcal{D})$ and arrows the elements of $\mathrm{Dbl}(\mathcal{D})$. These are indeed categories, since the definition of a double category lends associative and unital composition functions to both vertical arrows (whose domains and codomains are objects of $\mathcal{D}$) and double cells (whose horizontal domains and codomains are arrows in $\mathrm{Hor}(\mathcal{D})$). Having chosen our categories, we now define those functors required by categories internal to **Cat** :

- We first define the source and target functors, $s, t : D_1 \rightrightarrows D_0$: For any object (horizontal arrow) f in D_1, define $fs = f\mathrm{dom}$ and $ft = f\mathrm{cod}$. For any arrow (double cell) α in D_1, define αs to be the horizontal domain of α and αt to be the horizontal codomain of α. We check only the functoriality of s; that t is functorial is analogous. Objects in D_1 are horizontal arrows, the vertical domain of double cells, and we thus use vertical composition and identities of the double cells in D_1. We first recall that if we have two composable double cells $\alpha, \beta \in D_1$, then

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\big\downarrow\bullet & \alpha & \bullet\big\downarrow{\scriptstyle w} \\ C & \xrightarrow[g]{} & D \\ {\scriptstyle u'}\big\downarrow\bullet & \beta & \bullet\big\downarrow{\scriptstyle w'} \\ E & \xrightarrow[h]{} & F \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u\bullet u'}\big\downarrow\bullet & \alpha\bullet\beta & \bullet\big\downarrow{\scriptstyle w\bullet w'} \\ E & \xrightarrow[h]{} & F \end{array}$$

It is then clear that $\alpha s \bullet \beta s = u \bullet u' = (\alpha \bullet \beta)s$ and composition is thus preserved by s. For any horizontal arrow $f : A \to B$, we have the identity double cell

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle 1_A}\big\downarrow\bullet & 1_f & \bullet\big\downarrow{\scriptstyle 1_B} \\ A & \xrightarrow[f]{} & B \end{array}$$

It is again clear that $1_f s = 1_A = 1_{fs}$ and we have shown that s is functorial.

- We now define the identity-assigning functor, $e : D_0 \to D_1$: For any object A in D_0, define $Ae = \mathrm{id}_A$, the horizontal identity of A. For any (vertical) arrow $u : A \overset{\bullet}{\longrightarrow} B$, define $ue = \mathrm{id}_u$, the horizontal identity double cell induced by u. We now check that e is functorial. If we have a pair of composable arrows $A \overset{u}{\longrightarrow\!\!\bullet\!\!\longrightarrow} B \overset{u'}{\longrightarrow\!\!\bullet\!\!\longrightarrow} C$, consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ {\scriptstyle u}\bullet\!\downarrow & \mathrm{id}_u & \downarrow\!\bullet{\scriptstyle u} \\ B & \xrightarrow{\mathrm{id}_B} & B \\ {\scriptstyle u'}\bullet\!\downarrow & \mathrm{id}_{u'} & \downarrow\!\bullet{\scriptstyle u'} \\ C & \xrightarrow[\mathrm{id}_C]{} & C \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ {\scriptstyle u\bullet u'}\bullet\!\downarrow & \mathrm{id}_u \bullet \mathrm{id}_{u'} & \downarrow\!\bullet{\scriptstyle u\bullet u'} \\ C & \xrightarrow[\mathrm{id}_C]{} & C \end{array}$$

We note that the double cell $\mathrm{id}_u \bullet \mathrm{id}_{u'}$ is equal to the double cell $\mathrm{id}_{u\bullet u'}$, due to its boundary arrows. Then $(u{\bullet}u')e = \mathrm{id}_{u\bullet u'} = \mathrm{id}_u {\bullet} \mathrm{id}_{u'} = ue{\bullet}u'e$ and thus composition is preserved by e. If A is any object of D_0, we consider the diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ {\scriptstyle 1_A}\bullet\!\downarrow & \mathrm{id}_{1_A} & \downarrow\!\bullet{\scriptstyle 1_A} \\ A & \xrightarrow[\mathrm{id}_A]{} & A \end{array}$$

We note that the double cell id_{1_A} is equal to the double cell 1_{id_A}, due to its boundary arrow. Then $1_A e = \mathrm{id}_{1_A} = 1_{\mathrm{id}_A} = 1_{Ae}$ and thus identities are preserved by e and we have shown e to be functorial.

– Finally, we define the composition functor, $c : D_1 \times_{D_0} D_1 \to D_1$: For any pair of horizontal arrows $(f, g) \in D_1 \times_{D_0} D_1$, define $(f, g)c = f \circ g$. This is well defined, since $(f, g) \in D_1 \times_{D_0} D_1$ means that $ft = gs$ and thus are composable in $\mathrm{Hor}(\mathcal{D})$. For any two double cells $\alpha, \beta \in D_1 \times_{D_0} D_1$, define $(\alpha, \beta)c = \alpha \circ \beta$. Again, that $(\alpha, \beta) \in D_1 \times_{D_0} D_1$ implies that $\alpha t = \beta s$ and thus the horizontal domains and codomains match and this horizontal composite makes sense. We now verify that c is indeed functorial. Recall that composition within D_1 is vertical composition of double cells. If we have a vertically composable pair of pairs of horizontal arrows $(\alpha, \beta), (\gamma, \delta) \in D_1 \times_{D_0} D_1$, note that by the definition of composition of ordered pairs and finally by the middle-four interchange property inherited from $\mathcal{D}$, we have preservation of composition since

$$\begin{aligned}((\alpha,\beta)\bullet(\gamma,\delta))\,c &= (\alpha\bullet\gamma,\beta\bullet\delta)c\\ &= (\alpha\bullet\gamma)\circ(\beta\bullet\delta)\\ &= (\alpha\circ\beta)\bullet(\gamma\circ\delta)\\ &= (\alpha,\beta)c\bullet(\gamma,\delta)c\end{aligned}$$

To check that identities must commute, we first consider what types of identities must be in these ordered pairs. Note that since any pair $(f,g)\in D_1\times_{D_0}D_1$ must be such that the domain of g is the codomain of f and that any vertical identity is uniquely determined by its domain (or codomain), it follows that any pair of identities in $D_1\times_{D_0}D_1$ are actually pairs of the same identity. Furthermore, since we are composing these vertical identities horizontally, it follows that these domains must be endomorphisms. Otherwise, we would not be able to compose the cells. Visually, we have composites such as

$$\begin{array}{ccccc} A & \xrightarrow{f} & A & \xrightarrow{f} & A \\ {\scriptstyle 1_A}\big\downarrow\!\bullet & 1_f & {\scriptstyle 1_A}\big\downarrow\!\bullet & 1_f & \bullet\!\big\downarrow{\scriptstyle 1_A} \\ A & \xrightarrow{f} & A & \xrightarrow{f} & A \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{f\circ f} & A \\ {\scriptstyle 1_A}\big\downarrow\!\bullet & 1_f\circ 1_f & \bullet\!\big\downarrow{\scriptstyle 1_A} \\ A & \xrightarrow{f\circ f} & A \end{array}$$

It is clear, then, that $(1_f,1_f)c=1_{(f,f)c}$ and c is thus functorial. We now check that the required diagrams commute:

(1) Commutativity of source and target with composition: We check only commutativity of source. Target follows analogously. For any object $x\in D_0$, we have that $xes=\mathrm{id}_x s=x$. For any arrow $f:x\longrightarrow\!\!\!\!\!\!\bullet\;\;\, y\in D_0$ we have that $fes=\mathrm{id}_f s=f$.

(2) Commutativity of source and target with composition: Again, we check only the commutativity of source. For any pair of objects (horizontal arrows) $(f,g)\in D_1\times_{D_0}D_1$, we have that both $(f,g)cs=(f\circ g)s=fs$ and $(f,g)\pi_1 s=fs$. For any pair of arrows (double cells) $(\alpha,\beta)\in D_1\times_{D_0}D_1$, we also have that both $(\alpha,\beta)cs=(\alpha\circ\beta)s=\alpha s$ and $(\alpha,\beta)\pi_1 s=\alpha s$.

(3) Associativity of composition is inherited from the composition in Hor($\mathcal{D}$), since its components are both defined via this composition.

(4) Commutativity of left and right identities for composition: We check only the left identities. The commutativity of the right identities follows analogously. For any object pair $(x, f) \in D_0 \times_{D_0} D_1$, we know that this implies that $x = fs$. Then $(x, f)(e \times_{D_0} \mathrm{id}D_1)c = (\mathrm{id}_x, f)c = \mathrm{id}_x \circ f = f = \pi_2(x, f)$. For any arrow pair $(u, \alpha) \in D_0 \times_{D_0} D_1$, we know that this implies that $u = \alpha s$. Then $(u, \alpha)(e \times_{D_0} \mathrm{id}D_1)c = (\mathrm{id}_u, \alpha) = id_u \circ \alpha = \alpha$.

Having defined all of the appropriate functors such that all required diagrams commute, we conclude that $\mathcal{D}$ is indeed a category internal to **Cat**.

Conversely, suppose that we are given a category (D_0, D_1) internal to **Cat** with its accompanying data, namely those functors s, t, e and c that satisfy the commutativity of all the required diagrams. We define a double category $\mathcal{D}$ as the double category with objects those of D_0, vertical arrows the arrows of D_0, horizontal arrows the objects of D_1 and double cells the arrows of D_1. We define, for any two vertical arrows $u, v \in \mathrm{Ver}(\mathcal{D})$ with $ut = vs$, the vertical composite $u \bullet v$ simply as the composition of arrows in D_0. This composition then meets all of the associativity and unital requirements. We define, for any two horizontal arrows $f, g \in \mathrm{Hor}(\mathcal{D})$ with $ft = gs$, the horizontal composite $f \circ g$ as $(f, g)c$, the object function of the provided composition functor c. Again, because c comes from the internal category, all associativity and unital requirements are met. On double cells α and β, there is some variety. First, since D_1 is itself a category, it comes with its own associative and unital composition of double cells whose vertical domains and codomains (horizontal arrows) match. We can then define the vertical composite $\alpha \bullet \beta$ as this composite whenever $\alpha : f \to g$ and $\beta : g \to h$. On the other hand, we have our composition functor $c : D_1 \times_{D_0} D_1 \to D_1$, which is defined whenever $(\alpha, \beta) \in D_1 \times_{D_0} D_1$, or when $\alpha t = \beta s$ (which are vertical arrows, the horizontal domain and codomain). In such a case, then, we define the horizontal composite $\alpha \circ \beta$ as $(\alpha, \beta)c$, the arrow part of the composition functor. These compositions are both unital and associative. It remains, however, to show they satisfy the middle-four interchange. Indeed, this follows immediately from the definition of a product category and the functoriality of c. Suppose that we have double cells α, β, γ and δ such the composite $(\alpha \circ \beta) \bullet (\gamma \circ \delta)$ exists. Then

$$\begin{aligned}(\alpha \circ \beta) \bullet (\gamma \circ \delta) &= (\alpha, \beta)c \bullet (\gamma, \delta)c \\ &= [(\alpha, \beta) \bullet (\gamma, \delta)]\, c \text{ (functoriality of } c) \\ &= (\alpha \bullet \gamma, \beta \bullet \delta)c \\ &= (\alpha \bullet \gamma) \circ (\beta \bullet \delta)\end{aligned}$$

Having defined all parts of a double category and having verified that middle-four is satisfied, we can conclude that $\mathcal{D}$ as defined is indeed a double category and the proposition is proved. □

5.4 Double Inverse Semigroups Revisited

Double semigroups and double categories seem to share some properties. For example, both structures have two associative products defined on some of their data (double semigroups the semigroup products on their elements and double categories the horizontal and vertical compositions on their arrows and double cells). Though seemingly arbitrary at first glance, the choice that double semigroups must satisfy the interchange law is a choice which, when manipulated, yields some interesting results. Recall that the horizontal and vertical composition of double cells of a double category must satisfy a similar interchange law. Further evidence, then, that this interchange law may be the correct choice of relationship between the two associative binary operations of a double semigroup is that any double semigroup can be considered exactly as a double semicategory, where a double semicategory is defined as follows:

Definition 5.4.1. A *double semicategory* is a double category without the requirement that horizontal and vertical composition of vertical and horizontal arrows (and, consequently, double cells) have identities. ■

Note. It is interesting to note that a double semicategory is a semicategory internal to the category of semicategories.

Given a double semigroup, we can construct a double semicategory in the following way:

Construction 5.4.2. Let $(D, \circledcirc, \odot)$ be a double semigroup. Define $\mathbf{DCat}(D)$ to be the double semicategory with the following data:

– There is only one object, call it $*$.

– There is only one vertical arrow, call it v.

- With having only one vertical arrow, the vertical composition $\bullet$ is such that v is an identity. That is, we know that $v \bullet v = v$.

– There is only one horizontal arrow, call it h.

- It is noted now that the horizontal composition is such that h is an identity, as above.

– A square

$$\begin{array}{ccc} * & \xrightarrow{h} & * \\ {\scriptstyle v}\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow{\scriptstyle v} \\ * & \xrightarrow[h]{} & * \end{array}$$

is a double cell in $\mathbf{DCat}(D)$ if and only if α is an element of D.

- The vertical composition is $\odot$ and the horizontal composition is $\circledcirc$. Since $\circledcirc$ and $\odot$ are double semigroup operations and thus satisfy the middle four interchange, it follows that the horizontal and vertical composition of the double cells satisfy the interchange law, too. ♢

This construction is nice in the sense that it gives a double categorical interpretation of a double semigroup. It is, however, the case that this construct yields only double semigcategories. Though the fact that double semigroups satisfy the middle-four interchange and the operations are associative does not imply a stronger relationship between double semigroups and double categories, it would be nice to get one.

There is a construction of double semigroups given a double semicategory with one object, one vertical arrow and one horizontal arrow. It is worth noting it here:

Construction 5.4.3. Given a double semicategory D with one object, one idempotent vertical arrow v with respect to $\bullet$ and one idempotent horizontal arrow h with respect to $\circ$. construct a double semigroup $(S, \circledcirc, \odot)$ with elements being the double cells of D, $\circledcirc$ being $\circ$ and $\odot$ being $\bullet$. Since all double cells are composable in a single-object, idempotent-arrow double category, and the operations come from a double semicategory, middle-four is satisfied. The products are compositions and are obviously associative, so S is a double semigroup. $\diamondsuit$

We turn our attention now to attempting a construction of double semigroups given double categories with multiple objects. Our double semigroups of interest, in particular, are double inverse semigroups, so restricting ourselves to double inverse semigroups will have no drawbacks. Lawson [11] introduces a construction of inverse semigroups from groupoids:

Definition 5.4.4. A *groupoid* is a category in which every morphism is an isomorphism. ■

The construction is as follows:

Construction 5.4.5. Given a groupoid G, construct an inverse semigroup $(S, \odot)$ with elements the arrows of G and for any $a, b \in S$, the product is defined as

$$a \odot b = \begin{cases} a \bullet b & \text{if } a, b \in G_1 \text{ and } a\text{cod} = b\text{dom} \\ 0 & \text{otherwise} \end{cases}$$

This product is associative: for any 3-fold product $a \odot b \odot c$ in S, we have two possibilities. If each of a, b and c are composable arrows in G, then this product is simply the composite and is thus associative. If any of a, b or c are not composable or are not arrows in G, then we will have 0 in the resulting product. Then the whole product will be zero and associativity is trivial. This product also admits unique inverses: If a is an arrow in G, then $a^{\odot} = a^{-1}$, the unique inverse from G. The inverse of 0. If $a = 0^{\odot}$, then $a = 0a0 = 0$. ♢

Lawson also gives a construction of groupoids from inverse semigroups. This construction is exactly that in the next chapter, without the definition of (co)restrictions. Lawson remarks that the only inverse semigroups that arise from this construction are *primitive*; that is, if $0 \neq e \leq f$ are idempotents, then $e = f$. In an attempt to generalise this to double categories, we propose the following construction:

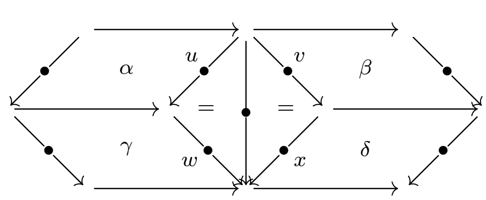
Construction 5.4.6. Given a double groupoid D, construct a double inverse semigroup $(S, \circledcirc, \odot)$ with elements the double cells of D and for any $a, b \in S$, products

$$a \circledcirc b = \begin{cases} a \circ b & \text{if } a, b \in \mathrm{Ver}(D) \text{ and } a\mathrm{hcod} = b\mathrm{hdom} \\ 0 & \text{otherwise} \end{cases}$$

and

$$a \odot b = \begin{cases} a \bullet b & \text{if } a, b \in \mathrm{Hor}(D) \text{ and } a\mathrm{vcod} = b\mathrm{vdom} \\ 0 & \text{otherwise} \end{cases}$$ ♢

That is, we can construct a double semigroup with zero, making all undefined horizontal and vertical composites of double cells zero, out of any double groupoid. This class of semigroups is restrictive, however, since only certain classes of double groupoids give rise to a double semigroup in this way. The problem comes from middle-four not always being satisfied in the resulting double semigroup. Suppose, for example, that one has the following configuration of double cells in a double groupoid:



In such a diagram, neither of the horizontal composites, $\alpha \circ \beta$ and $\gamma \circ \delta$, exist and are thus zero in the double semigroup we are constructing. Let us, for convenience, use the same symbols for the semigroup operations as the composition in the double category. Then we have that $(\alpha \circ \beta) \bullet (\gamma \circ \delta) = 0 \bullet 0 = 0$. However, we know that the vertical composites $\alpha \bullet \gamma$ and $\beta \bullet \delta$ are both non-zero (their domains and codomains match) and the composites are horizontally composable (since the composites of the horizontal domains and codomains are equal by assumption). So the composite $(\alpha \bullet \gamma) \circ (\beta \bullet \delta)$ is not zero and middle-four is not satisfied. One can solve this problem by requiring unique factorisation of both the vertical and horizontal arrows in a double category. That is, the requirement that two composites $f \bullet g = f' \bullet g'$ implies that $f = f'$ and $g = g'$ (similarly for the horizontal composition). If G is such a double groupoid, a an object and $f : a \multimap\!\!\!\!\bullet\; b$ a vertical arrow, then

$$f = 1_a \bullet f = f \bullet 1_b$$

implies, by unique factorisation, that $f = 1_a = 1_b$. That is, there are only identity arrows in a double groupoid with unique factorisation. The corresponding double inverse semigroup, then, is a disjoint union of double groups, or Abelian groups. The correspondence, then, between double groupoids and double inverse semigroups induced by this construction is not very interesting.

To be able to construct a non-trivial double inverse semigroup from *any* double groupoid, then, does not appear to be possible. We explore, then, another construction of inverse semigroups from another class of groupoids, which will be detailed in the next chapter, that will solve this problem.

Chapter 6

Inductive Groupoids

The troubles encountered in the previous chapter while constructing double semigroups from double semicategories have not been much studied. However, there has been some work done constructing certain categories from certain semigroups. In particular, Lawson [11] studied the relationship between inverse semigroups and inductive groupoids. Unless commonly known or explicitly stated, it is from Lawson that the following definitions and theorems have been retrieved. In this chapter, we will consider a special type – inductive – of groupoids. In order to be able to define an inductive groupoid, we will require first the notion of an ordered groupoid. An ordered groupoid in turn requires the definition of some kind of order on the groupoid structure, namely that of a partial order:

Definition 6.0.7. Let S be a set. A *partial order on* S, $\leq$, is a binary relation such that, for all $a, b, c \in S$, $\leq$ is

(i) reflexive : $a \leq a$.

(ii) antisymmetric : if $a \leq b$ and $b \leq a$, then $a = b$.

(iii) transitive : if $a \leq b$ and $b \leq c$, then $a \leq c$.

A set S equipped with a partial order is called a *partially ordered set.* ■

Definition 6.0.8. A partially ordered set S is said to be a *meet-semilattice* if, for all $a, b \in S$, the greatest lower bound, called the meet, of the set $\{a, b\}$ exists. ■

6.1 Inductive Groupoids

In any category, there is an obvious correspondence between the objects and identity morphisms given by $1_a \mapsto a$ and $a \mapsto 1_a$. It is due to this correspondence that we identify, without any ambiguity, the set of objects in a given category with its set of identity morphisms. In the following definition, for example, we will use this identification to discuss comparisons between objects in a category using the partial order on its arrows. In addition to this identification allowing the objects of a groupoid to inherit the order structure on the arrows, it also admits a more streamlined notation when discussing partial orders on categorical structures such as in the following definition:

Definition 6.1.1. Let $(G, \bullet)$ be a groupoid and let $\leq$ be a partial order defined on the arrows of G. We call $(G, \bullet, \leq)$ an *ordered groupoid* whenever the following conditions are satisfied (where G_1 is the set of arrows in G) :

(i) For all $x, y \in G_1$, $x \leq y$ implies $x^{-1} \leq y^{-1}$.

(ii) For all $x, y, u, v \in G_1$, if $x \leq y$, $u \leq v$ and the composites xu and yv exist in G, then $xu \leq yv$.

(iii) Let $x \in G_1$ and let e be an object in G such that $e \leq x\text{dom}$. Then there is a unique element $(e_*|x) \in G_1$, called the restriction of x by e, such that $(e_*|x) \leq x$ and $(e_*|x)\text{dom} = e$.

(iv) Let $x \in G_1$ and let e be an object in G such that $e \leq x\text{cod}$. Then there is a unique element $(x|_*e) \in G_1$, called the corestriction of x by e, such that $(x|_*e) \leq x$ and $(x|_*e)\text{cod} = e$.

We say that G is an *inductive groupoid* if the further condition that the objects of G (or, equivalently by our identification, the arrows of G) form a meet-semilattice is satisfied. ■

We now describe maps between inductive groupoids:

Definition 6.1.2. A functor $F : G \to G'$ between two inductive groupoids is called *inductive* if it preserves both the order and the meet operation on the set of objects. ■

Proposition 6.1.3. *Let $(G, \bullet, \leq)$ be an ordered groupoid. Let $x, y, z \in G_1$ and let e and f be objects in G.*

(1) If $x \leq y$, then $x\mathrm{dom} \leq y\mathrm{dom}$ and $x\mathrm{cod} \leq y\mathrm{cod}$.

(2) If the composite xy exists and $e \leq (xy)\mathrm{dom}$, then

$$(e_*|xy) = (e_*|x)((e_*|x)\mathrm{cod}_*|y).$$

(3) If the composite xy exists and $e \leq (xy)\mathrm{cod}$, then

$$(xy|_*e) = (x|_*(y|_*e)\mathrm{dom})(y|_*e).$$

(4) If $z \leq xy$, then there exist $x', y' \in G_1$ such that $x'y'$ exists, $x' \leq x$, $y' \leq y$ and $z = x'y'$.

(5) If $f \leq e \leq x\mathrm{dom}$, then $(f_*|x) \leq (e_*|x) \leq x$.

(6) If $f \leq e \leq x\mathrm{cod}$, then $(x|_*f) \leq (x|_*e) \leq x$.

Proof. (1): If $x \leq y$, then $x^{-1} \leq y^{-1}$ by the first axiom of ordered groupoids. Then $x\mathrm{dom} = xx^{-1} \leq yy^{-1} = y\mathrm{dom}$ and $x\mathrm{cod} = x^{-1}x \leq y^{-1}y = y\mathrm{cod}$ by the second axiom of inductive groupoids.

(2): We have that $e \leq (xy)\mathrm{dom} = x\mathrm{dom}$, so the restriction $(e_*|x)$ exists. Also, $(e_*|x) \leq x$ so $(e_*|x)\mathrm{cod} \leq x\mathrm{cod} = y\mathrm{dom}$ and thus the restriction $((e_*|x)\mathrm{cod}_*|y) \leq y$ exists. Then by the axioms of ordered groupoids, $(x|_*e)((x|_*e)\mathrm{cod}_*|y) \leq xy$ and thus $((e_*|x)((e_*|x)\mathrm{cod}_*|y))\mathrm{dom} = e \leq (xy)\mathrm{dom}$ and thus, by uniqueness of restrictions,

$$(e_*|x)((e_*|x)\mathrm{cod}_*|y) = (e_*|xy).$$

(3): This is analogous to (2).

(4): Let $z \leq xy$. Then by the axioms of ordered groupoids, $z\mathrm{cod} \leq (xy)\mathrm{cod}$ and thus the corestriction $(xy|_* z\mathrm{cod})$ exists. By the definition of corestriction, $(xy|_* z\mathrm{cod})\mathrm{cod} = z\mathrm{cod}$ and $(xy|_* z\mathrm{cod}) \leq xy$. Then by the uniqueness of corestrictions, $z = (xy|_* z\mathrm{cod})$. By (3) of this proposition, then,

$$z = (xy|_* z\mathrm{cod}) = (x|_*(y|_* z\mathrm{cod})\mathrm{dom})(y|_* z\mathrm{cod}).$$

Let $x' = (x|_*(y|_* z\mathrm{cod})\mathrm{dom}) \leq x$ and let $y' = (y|_* z\mathrm{cod}) \leq y$. Then $z = x'y'$ and the remaining requirements are satisfied.

(5) : Suppose that $e \leq f \leq x\mathrm{dom}$. Then the restrictions $(f_*|x)$ and $(e_*|x)$ certainly both exist and are both less than or equal to x. We show, then, that $(f_*|x) \leq (e_*|x)$. First, note that $f \leq e = (e_*|x)\mathrm{dom}$ and thus the restriction $(f_*|(e_*|x))$ exists. However, $(f_*|x)\mathrm{dom} = f$ and $(f_*|x) \leq f$, so by the uniqueness of restrictions, $(f_*|x) = (f_*|(e_*|x)) \leq (e_*|x)$.

(6): This is analogous to (5). □

6.2 Inductive Groupoids to Inverse Semigroups

Having established the definition of an inductive groupoid and having proved some elementary yet crucial facts about the order structures on them, we can now attempt to construct inverse semigroup from inductive groupoids. This construction comes directly from Lawson [11].

Construction 6.2.1. Given an inductive groupoid $(G, \bullet, \leq, \wedge)$, construct an inverse semigroup $(S, \odot)$ with $S = G_1$ and, for any $a, b \in S$,

$$a \odot b = (a|_* a\mathrm{cod} \wedge b\mathrm{dom}) \bullet (a\mathrm{cod} \wedge b\mathrm{dom}_*|b).$$

Since, for any $a \in S = G_1$, $a\mathrm{cod}$ and $a\mathrm{dom}$ are objects in G, this product is always defined, since G_0 is a meet-semilattice with respect to the $\wedge$ operation used. ♢

The existence of the product is not enough to conclude that S is an inverse semigroup. It remains to be shown that $\odot$ is associative and has unique inverses. Showing associativity requires the following lemma:

Lemma 6.2.2. *Let* $(G, \bullet, \leq, \wedge)$ *be an inductive groupoid and define, for every pair of arrows* $x, y \in G$,

$$\langle x, y\rangle = \{(x', y')|x'\mathrm{cod} = y'\mathrm{dom}, x' \leq x, y' \leq y\}.$$

Then there is a maximal element $(x', y') \in \langle x, y\rangle$ *(with respect to the pairwise ordering) and* $x \odot y = x' \bullet y'$.

Proof. We note that since G is an inductive groupoid, G_0 is a meet-semilattice and thus, for all $x, y \in G_1$, the meet $e = x\mathrm{cod} \wedge y\mathrm{dom}$ exists. Then both the restriction $(e_*|y)$ and the corestriction $(x|_*e)$ exist with, of course, $(x|_*e) \leq x$ and $(e_*|y) \leq y$. Therefore, $((x|_*e), (e_*|y)) \in \langle x, y\rangle$. We want to show that this is the maximal element. If this is true, the coincidence of the two operations follows by definition of $\odot$.

Let $(u, v) \in \langle x, y\rangle$. Then $u\mathrm{cod} = v\mathrm{dom} = f$ and both $u \leq x$ and $v \leq y$. Then by Proposition 6.1.3(1), it is true that $u\mathrm{cod} \leq x\mathrm{cod}$ and thus by the uniqueness of corestrictions in an inductive groupoid, $u = (x|_*u\mathrm{cod}) = (x|_*f)$. Similarly, $v = (f_*|y)$. We note now that $f \leq u\mathrm{cod}, v\mathrm{dom} \leq x\mathrm{cod}, y\mathrm{dom}$ and thus $f \leq e$. Therefore, by Proposition 6.1.3(5) and (6), $u = (x|_*f) \leq (x|_*e)$ and $v \leq (e_*|x)$ and we have shown that $(u, v) \leq ((x|_*e), (e_*|x))$, as required. □

We can now finally prove that the product defined from inductive groupoids is indeed associative:

Lemma 6.2.3. *Let* G *be an inductive groupoid. For all* $x, y, x \in G_1$, $x \odot (y \odot z) = (x \odot y) \odot z$.

Proof. To prove that $x \odot (y \odot z) = (x \odot y) \odot z$ requires that one shows that both $x \odot (y \odot z) \leq (x \odot y) \odot z$ and $x \odot (y \odot z) \geq (x \odot y) \odot z$, by the antisymmetry of $\leq$. We prove only one direction of this; the other direction follows analogously.

Let us start by letting $(x \odot y) \odot z = a \bullet z'$, where $(a, z') \in \langle x \odot y, z\rangle$ is maximal. This is justified by the the above lemma. We can then also choose $a = x' \bullet y'$, where

$(x', y') \in \langle x, y \rangle$ is maximal. Collectively, then, by the definition of $\langle x \odot y, z \rangle$ and $\langle x, y \rangle$, we have that $a \leq x \odot y = x' \bullet y'$, $z' \leq z$, $x' \leq x$ and $y' \leq y$. Since $a \leq x' \bullet y'$, we can use Proposition 6.1.3(4) to conclude that there are elements $x'' \leq x'$ and $y'' \leq y'$ such that $a = x'' \bullet y''$. Then, by substitution and associativity of $\bullet$, we have

$$(x \odot y) \odot z = a \bullet z' = (x'' \bullet y'') \bullet z' = x'' \bullet (y'' \bullet z').$$

By transitivity of $\leq$, $y'' \leq y$ and $z' \leq z$, so $(y'', z') \in \langle y, z \rangle$. This implies that $y'' \circ z' \leq y \odot z$. Combine this with the fact that $x'' \leq x$, and we conclude that $(x'', y'' \bullet z') \in \langle x, y \odot z \rangle$, or that $(x \odot y) \odot z = x'' \circ (y'' \circ z') \leq x \odot (y \odot z)$. □

Finally, we can now show that our construction yields an inverse semigroup and thus satisfies our needs.

Theorem 6.2.4. *For any inductive groupoid G, $IS(G)$ as defined in Construction 6.2.1 is an inverse semigroup.*

Proof. We have already shown that $IS(G)$ is a semigroup by verifying that its product is associative. Clearly, every element $s \in IS(G)$ has at least one inverse, namely $s' = s^{-1}$, the inverse from the groupoid. This follows from noting that both $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s$ and that the operations are the same whenever these composites are defined. We now check that all idempotents commute. The idempotents of $IS(G)$ are indeed the objects of G and thus, for any idempotents $e, f \in IS(G)$,

$$e \odot f = (e|_* e \wedge f)(e \wedge f_*|f) = e \wedge f = (f|_* e \wedge f)(e \wedge f_*|e) = f \odot e.$$

Note that the equality $(e_*|e \wedge f) = e \wedge f$ holds by the uniqueness of restrictions: since $e \leq e \wedge f$, we know that e is idempotent and thus $e = e\text{dom} \leq (e \wedge f)\text{dom}$. Therefore, by uniqueness of restrictions, $e \wedge f = (e_*|e \wedge f)$. By Lemma 2.2.3, then, we conclude that $IS(G)$ is indeed inverse. □

6.3 Inverse Semigroups to Inductive Groupoids

In addition to constructing inverse semigroups from inductive groupoids, Lawson [11] also details how to construct inductive groupoids from inverse semigroups:

Construction 6.3.1. Given an inverse semigroup $(S, \odot)$ with the natural partial ordering $\leq$, define a groupoid, $IG(S)$, with the following data:

– Its objects are the idempotents of S; $IG(S)_0 = E(S)$.

– Its arrows are the elements of S. For any $s \in S$, define $s\text{dom} = ss^{\odot}$ and $s\text{cod} = s^{\odot}s$, where $s^{\odot}$ is the inverse of s.

- For any $a, b \in IG(S)_1$, if $a\text{cod} = b\text{dom}$, define the composite $a \bullet b = a \odot b$, the product in S. This composition is well defined (i.e., the composite has the proper domain and codomain) and therefore inherently associative: if $a \bullet b$ is defined, then $a\text{cod} = b\text{dom}$, or $a^{\odot}a = bb^{\odot}$. Then $(a \bullet b)\text{dom} = (ab)\text{dom} = (ab)(ab)^{\odot} = abb^{\odot}a^{\odot} = aa^{\odot}aa^{\odot} = aa^{\odot} = a\text{dom}$. Similarly, we have $(a \bullet b)\text{cod} = b\text{cod}$.
- Since, for any $a \in S$, $(aa^{\odot})a = a(a^{\odot}a) = a$, this composition is always unital.
- It follows, then, that every arrow is an isomorphism with $a^{-1} = a^{\odot}$, since $a^{-1} \bullet a = a^{\odot}a = a\text{cod}$ and $a \bullet a^{-1} = aa^{\odot} = a\text{dom}$ (recall that we have identified the objects with identity arrows). $\diamondsuit$

Since the objects of S are the arrows of $IG(S)$, it follows that $\leq$ is also a partial order on the arrows of $IG(S)$. This has various consequences. In particular, we note the following:

Theorem 6.3.2. *$IG(S)$ is an inductive groupoid with, for all $a \in IG(S)$, $(a|_*e) = ae$ for all objects $e \leq a\text{cod}$ and $(e_*|a) = ea$ for all objects $e \leq a\text{dom}$.*

Proof. It is known by Construction 6.3.1 that $IG(S)$ is a groupoid. It remains only to check that it satisfies those conditions outlined in the definition of inductive groupoid. We do this now:

(i) If $x, y \in IG(S)_1$, then $x = ey$ for some $e \in E(S)$. Then $x^{-1} = y^{-1}e$ and $x^{-1} \leq y^{-1}$.

(ii) If $x, y, u, v \in IG(S)_1$ with $x \leq y$ and $u \leq v$ and the composites xu and xy existing. Then $x = ye$ for some idempotent e and $u = vf$ for some idempotent f. Then $xu = (ye)(vf) = x(ev)f$. However, $ev = vi$ for some idempotent i and thus $xu = y(vi)f = (yv)(if)$ and thus $xu \leq yv$.

(iii) Let $s \in IG(S)_1$ with an idempotent $e \leq s\text{cod} = s^{\odot}s$. Then it is obvious that $(s|_*e) = se \leq s$. Also $(s|_*e)\text{cod} = (se)\text{cod} = e\text{cod} = e^{\odot}e = e$. Now, for uniqueness. Let $b \leq (s|_*e)$ such that $b\text{cod} = bb^{\odot} = e$. Then by the definition of corestriction, $b = (b|_*ss^{\odot}) = bss^{\odot}$ and thus $b = (b|_*ss^{\odot}) = (s|_*e)$.

(iv) The proof of the restriction property follows analogously. □

6.4 An Isomorphism of Categories

Being able to construct an inductive groupoid given any inverse semigroup and vice-versa is extremely useful. One may naturally, however, wonder whether these constructions induce functors between the category of inductive groupoids with inductive functors and the category of inverse semigroups with semigroup homomorphisms. Lawson establishes and describes such functors and also proves that they form an isomorphism of categories between the category of inductive groupoids and inductive functors and the category of inverse semigroups and semigroup homomorphisms. To more clearly discuss these categories, we make some notational choice:

Notation. Denote the category of inverse semigroups and semigroup homomorphisms as **IS**. Denote the category of inductive groupoids and inductive functors as **IG**.

Now, we prove the following theorem, due to Lawson [11]:

Theorem 6.4.1. *The categories* **IG** *and* **IS** *are isomorphic.*

Proof. We define a pair of functors

$$\mathbf{IG} \underset{F'}{\overset{F}{\rightleftarrows}} \mathbf{IS}$$

as follows:

(i) $F : \mathbf{IG} \to \mathbf{IS}$:

- On objects: Given any inductive groupoid G, define $GF = IS(G)$, as defined in Construction 6.2.1. Recall that $IS(G) = G_1$ with product, for any $a, b \in IS(G)$, defined by

$$a \odot b = (a|_* a\text{cod} \wedge b\text{dom}) \bullet (a\text{cod} \wedge b\text{dom}_*|b).$$

- On arrows: For any inductive functor $f : G \to G'$, define $fF : GF \to GF' = f_1$, the arrow function of f. The arrow function preserves composition of arrows, which corresponds to multiplication of semigroup elements thus fF is a semigroup homomorphism. Since F is the arrow function of a functor, F is trivially functorial.

(ii) $F' : \mathbf{IS} \to \mathbf{IG}$:

- On objects: Given any inverse semigroup S, define $SF' = IG(S)$, as defined in Construction 6.3.1. Recall that $IG(S)$ has the following data:
  - Objects: $IG(S)_0 = E(S)$.
  - Arrows: $IG(S)_1 = S$.
- On Arrows: For any semigroup homomorphism $\varphi : S \to S'$, define $\varphi F' : SF' \to S'F'$ as the functor with object function φ restricted to $E(S)$ and arrow function φ. This is functorial since the composition in G_1 is the product in S and homomorphisms preserve products. This functor is also inductive since, for any arrow $a : aa^{\odot} \to a^{\odot}a$, the object and arrow functions are semigroup homomorphisms which imply that domains and

codomain are preserved. This preserves the order structure, then, and therefore preserves the meet structure.

We must now check that these two functors form an isomorphism of categories, by verifying that the composites FF' and $F'F$ are both identity functors.

Because the arrows of G are identified with the elements of $IS(G)$, any order on G will be the same as the order on the elements of $IS(G)$.

We also recall that, by definition, the composition in $IG(S)$ is exactly that of the product in S.

Because the orders are preserved and the products are the same, it is obvious that $SF'F = IS(IG)(S) = S$ for any inverse semigroup S. We now consider the composite $GFF' = IG(IS(G))$ for an inductive groupoid G. For any $a, b \in G$, we have that $a \odot b = (a|_* a\text{cod} \wedge b\text{dom})(a\text{cod} \wedge b\text{dom}_*|b)$ in $IS(G)$ and finally that $a \odot b = a(a^{\odot}abb^{\odot})b = ab$ in $IG(IS(G))$. That is, the products are exactly the same in $IG(IS(G))$ as they are in G and thus $IG(IS(G)) = G$. $\square$

We will now demonstrate this result by giving an example of an inverse semigroup and then calculating its corresponding inductive groupoid.

Example 6.4.2. Consider the following inverse semigroup $(S, \odot)$ (semigroup (5,415) of the `smallsemi` package of GAP [7]):

$\odot$	1	2	3	4	5
1	1	1	1	1	1
2	1	1	4	1	2
3	1	5	1	3	1
4	1	2	1	4	1
5	1	1	3	1	5

It is the case that S is the only non-commutative inverse semigroup of order 5 containing a non-idempotent element. We note that idempotents of S are $E(S) = \{1, 4, 5\}$.

It is routine to check that the inverses of each element in S are

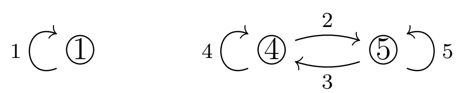
$$1^{\odot} = 1, 2^{\odot} = 3, 3^{\odot} = 2, 4^{\odot} = 4, \text{ and } 5^{\odot} = 5.$$

We note that $3\text{dom} = 3 \odot 3' = 3 \odot 2 = 5$, $3\text{cod} = 3' \odot 3 = 2 \odot 3 = 4$, $2\text{dom} = 4$, and $2\text{cod} = 5$. The associated inductive groupoid, then, has

– Objects: $\{1, 4, 5\}$

– (Non-identity) Arrows: $\{2 : 4 \to 5, 3 : 5 \to 4\}$

And has the visual appearance



We note that the order on this groupoid is trivial. ▲

Chapter 7

Double Inductive Groupoids

In this chapter, we will introduce the notion of a double inductive groupoid, a double categorical version of the inductive groupoids introduced in the preceding chapter. Much like the preceding chapter, we will define constructions of double inverse semigroups, and vice-versa, to establish an isomorphism between the category of double inductive groupoids with double inductive functors and the category of double inverse semigroups and double semigroup homomorphisms. We will then characterise double inverse semigroups using this isomorphism.

7.1 Double Inductive Groupoids

Recall that, in a category, there exists a one-to-one correspondence between the identity arrows and objects. In a double category, there are two such correspondences:

- Between the horizontal identity double cells of a double category and its vertical arrows.
- Between the vertical identity double cells of a double category and its horizontal arrows.

Recall that after introducing double categories, it was presented as an example that, given a double category, one can consider two natural categories: one of which having vertical arrows as objects with double cells as morphisms with horizontal composition and the other having horizontal arrows as objects with double cells as morphisms with vertical composition. It is through these two naturally induced categories that we will hereafter identify the vertical arrows of a double category with double cells that serve as identities for horizontal composition and identify the horizontal arrows with those double cells that serve as identities for vertical composition. We may now make the following definition:

Definition 7.1.1. A *double inductive groupoid*, denoted DIG,

$$\mathcal{G} = (\mathrm{Obj}(\mathcal{G}), \mathrm{Ver}(\mathcal{G}), \mathrm{Hor}(\mathcal{G}), \mathrm{Dbl}(\mathcal{G}), \leq, \lesssim)$$

is a double groupoid (i.e., a double category in which every vertical and horizontal arrow is an isomorphism and each double cell is an isomorphism with respect to both the horizontal and vertical composition) such that

(i) $(\mathrm{Ver}(\mathcal{G}), \mathrm{Dbl}(\mathcal{G}))$ is an inductive groupoid.

- We denote the composition in this inductive groupoid – the horizontal composition from $\mathrm{Dbl}(\mathcal{G})$ – with $\circ$. We denote the partial order on this groupoid as $\leq$. If e and f are horizontal identities (vertical arrows), we denote their meet as $e \wedge_h f$. For a cell $\alpha \in \mathrm{Dbl}(\mathcal{G})$ and a vertical arrow $e \in \mathrm{Ver}(\mathcal{G})$ such that $e \leq \alpha\mathrm{hdom}$, we denote the horizontal restriction of α by e by $(e_*|\alpha)$. Similarly, if e is a vertical arrow such that $e \leq \alpha\mathrm{hcod}$, we denote the horizontal corestriction of α by e by $(\alpha|_*e)$.

(ii) $(\mathrm{Hor}(\mathcal{G}), \mathrm{Dbl}(\mathcal{G}))$ is an inductive groupoid.

- We denote the composition in this inductive groupoid – the vertical composition from $\mathrm{Dbl}(\mathcal{G})$ – with $\bullet$. We denote the partial order on this groupoid as $\lesssim$. If e and f are vertical identities (horizontal arrows), we denote their meet as $e \wedge_v f$. For a cell $\alpha \in \mathrm{Dbl}(\mathcal{G})$ and a horizontal arrow $e \in \mathrm{Hor}(\mathcal{G})$ such that $e \lesssim \alpha\mathrm{vdom}$, we denote the horizontal restriction of α by e by $[e_*|\alpha]$. Similarly, if e is a horizontal arrow such that $e \lesssim \alpha\mathrm{vcod}$, we denote the horizontal corestriction of α by e by $[\alpha|_*e]$.

(iii) If a, b are double cells, f', g' are horizontal arrows and f, g are vertical arrows, the following laws about restrictions and corestrictions preserving composition hold:

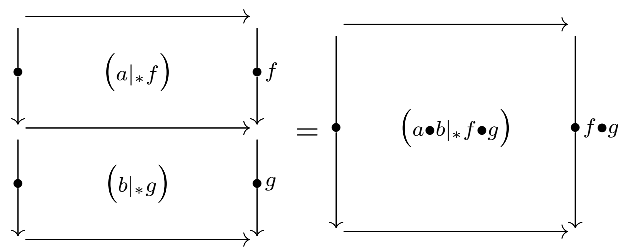
(a) $\big(a \bullet b|_* f \bullet g\big) = \big(a|_* f\big) \bullet \big(b|_* g\big)$.

(b) $\big[a \circ b|_* f' \circ g'\big] = \big[a|_* f'\big] \circ \big[b|_* g'\big]$.

(c) $\big(f \bullet g_*|a \bullet b\big) = \big(f_*|a\big) \bullet \big(g_*|b\big)$.

(d) $\big[f' \circ g'_*|a \circ b\big] = \big[f'_*|a\big] \circ \big[g_*|b\big]$.

The rule $\big(a \bullet b|_* f \bullet g\big) = \big(a|_* f\big) \bullet \big(b|_* g\big)$, visually:

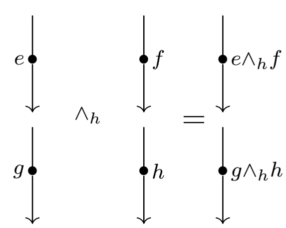


(iv) If e, f, g and h are horizontal arrows and e', f', g' and h' are vertical arrows, the following laws about composition preserving meets hold:

(a) $(e \wedge_v f) \circ (g \wedge_v h) = (e \circ g) \wedge_v (f \circ h)$.

(b) $(e' \wedge_h f') \bullet (g' \wedge_h h') = (e' \bullet g') \wedge_h (f' \bullet h')$.

The rule $(e \wedge_h f) \bullet (g \wedge_h h) = (e \bullet g) \wedge_h (f \bullet h)$, visually:



(v) If e and g are horizontal arrows, e' and g' are vertical arrows and each of f, h, f' and h' are objects, the following laws about restrictions and corestrictions preserving meets hold:

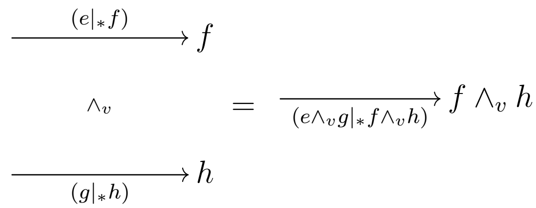
(a) $(e|_* f) \wedge_v (g|_* h) = (e \wedge_v g|_* f \wedge_v h)$.

(b) $[e'|_* f'] \wedge_h [g'|_* h'] = [e' \wedge_h g'|_* f' \wedge_h h']$.

(c) $(e_*|f) \wedge_v (g_*|h) = (e \wedge_v g_*|f \wedge_v h)$.

(d) $[e'_*|f'] \wedge_h [g'_*|h'] = [e' \wedge_h g'_*|f' \wedge_h h']$.

The rule $(e|_*f) \wedge_v (g|_*h) = (e \wedge_v g|_*f \wedge_v h)$, visually:



(vi) If a is a double cell, f a horizontal arrow, g a vertical arrow and x an object such that $f \lesssim a\text{vcod}$, $g \leq a\text{hcod}$ and $x = f\text{hcod} \wedge g\text{vcod}$, then the following law about commuting restrictions and corestrictions holds:

$$([a|_*f]|_*[g|_*x]) = [(a|_*g)|_*(f|_*x)]$$

With similar quantifications, the laws about commuting restrictions and corestrictions are:

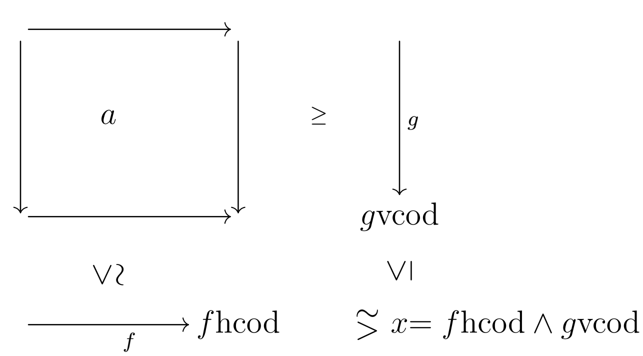
(a) $([a|_*f]|_*[g|_*x]) = [(a|_*g)|_*(f|_*x)]$.

(b) $[(a|_*g)|_*(f|_*x)] = ([a|_*f]|_*[g|_*x])$.

(c) $([x_*|g]_*|[f_*|a]) = [(x_*|f)_*|(g_*|a)]$.

(d) $[(x_*|f)_*|(g_*|a)] = ([x_*|g]|[f_*|a])$.

The rule $([a|_*f]|_*[g|_*x]) = [(a|_*g)|_*(f|_*x)$, visually:



(vii) If e, f, g and h are vertical and horizontal arrows, the following law about commuting meets holds:

$$(e \wedge_h f) \wedge_v (g \wedge_h h) = (e \wedge_v g) \wedge_h (f \wedge_v h).$$

Note that this is exactly saying that vertical and horizontal meets satisfy middle-four interchange.

(viii) If e, f are vertical arrows and e', f' are horizontal arrows, the following laws about domains and codomains preserving meets hold:

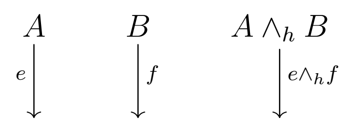
(a) $(e \wedge_h f)\mathrm{vdom} = e\mathrm{vdom} \wedge_h f\mathrm{vdom}$.

(b) $(e \wedge_h f)\mathrm{vcod} = e\mathrm{vcod} \wedge_h f\mathrm{vcod}$.

(c) $(e' \wedge_v f')\mathrm{hdom} = e'\mathrm{hdom} \wedge_v f'\mathrm{hdom}$.

(d) $(e' \wedge_v f')\mathrm{hcod} = e'\mathrm{hcod} \wedge_v f'\mathrm{hcod}$.

The rule $(e \wedge_h f)\mathrm{vdom} = e\mathrm{vdom} \wedge_h f\mathrm{vdom}$, visually:



(ix) If a is a double cell, e a vertical arrow and e' a horizontal arrow, then the following laws about domains and codomains preserving restrictions and corestrictions hold:

(a) $(a|_* e)\mathrm{vdom} = (a\mathrm{vdom}|_* e\mathrm{vdom})$.

(b) $(a|_* e)\mathrm{vcod} = (a\mathrm{vcod}|_* e\mathrm{vcod})$.

(c) $(e_*|a)\mathrm{vdom} = (e\mathrm{vdom}_*|a\mathrm{vdom})$.

(d) $(e_*|a)\mathrm{vcod} = (e\mathrm{vcod}_*|a\mathrm{vcod})$.

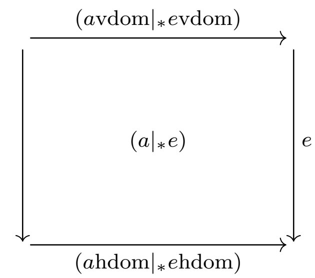
(e) $[a|_* e']\mathrm{hdom} = [a\mathrm{hdom}|_* e'\mathrm{hdom}]$.

(f) $[a|_* e']\mathrm{hcod} = [a\mathrm{hcod}|_* e'\mathrm{hcod}]$.

(g) $[e'_*|a]\text{hdom} = [e'\text{vdom}_*|a\text{hdom}]$.

(h) $[e'_*|a]\text{hcod} = [e'\text{hcod}_*|a\text{hcod}]$.

The rules $(a|_*e)\text{vdom} = (a\text{vdom}|_*e\text{vdom})$ and $(a|_*e)\text{hdom} = (a\text{hdom}|_*e\text{hdom})$, visually:



■

7.2 Double Inductive Groupoids to Double Inverse Semigroups

Construction 7.2.1. Given a double inductive groupoid

$$\mathcal{G} = (\text{Obj}(\mathcal{G}), \text{Ver}(\mathcal{G}), \text{Hor}(\mathcal{G}), \text{Dbl}(\mathcal{G})),$$

we construct a double inverse semigroup $\text{DIS}(\mathcal{G}) = (S, \circledcirc, \odot)$ as follows:

– Its elements are the double cells of $\mathcal{G}$; $S = \text{Dbl}(\mathcal{G})$.

– For any $a, b \in S$, define

$$a \circledcirc b = (a|_* a\text{hcod} \wedge_h b\text{hdom}) \circ (a\text{hcod} \wedge_h b\text{hdom}_*|b)$$

– For any $a, b \in S$, define

$$a \odot b = [a|_* a\text{vcod} \wedge_v b\text{vdom}] \bullet [a\text{vcod} \wedge_v b\text{vdom}_*|b] \qquad \diamondsuit$$

Note that these are indeed inverse semigroup operations, by Theorem 6.2.4. It remains, however, to show that middle-four is satisfied. We now verify that this is the case, first employing the definitions of $\circledcirc$ and $\odot$:

$$\begin{aligned}
&(a \circledcirc b) \odot (c \circledcirc d) \\
&=_1 \Big[a \circledcirc b\Big|_*(a \circledcirc b)\mathrm{vcod} \wedge_v (c \circledcirc d)\mathrm{vdom}\Big] \bullet \Big[(a \circledcirc b)\mathrm{vcod} \wedge_v (c \circledcirc d)\mathrm{vdom}_*\Big|c \circledcirc d\Big] \\
&=_2 \Big[(a|_* a\mathrm{hcod} \wedge_h b\mathrm{hdom}) \circ (a\mathrm{hcod} \wedge_h b\mathrm{hdom}_*|b)\Big|_*(a \circledcirc b)\mathrm{vcod} \wedge_v (c \circledcirc d)\mathrm{vdom}\Big] \\
&\quad \bullet \Big[(a \circledcirc b)\mathrm{vcod} \wedge_v (c \circledcirc d)\mathrm{vdom}_*\Big|(c|_* c\mathrm{hcod} \wedge_h d\mathrm{hdom}) \circ (c\mathrm{hcod} \wedge_h d\mathrm{hdom}_*|d)\Big]
\end{aligned}$$

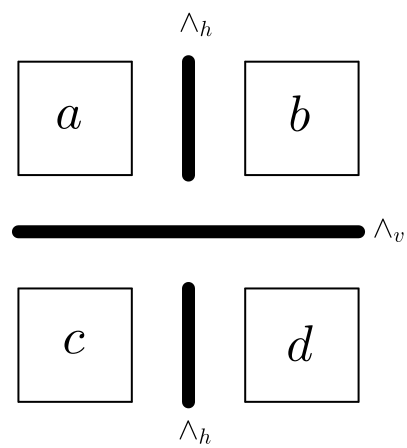
We can now apply the definition of $\circledcirc$, the fact that (co)domains preserve composition in a double groupoid and then, finally, Axiom (iv) to manipulate the (duplicate inner) identities in the above (co)restrictions:

$$\begin{aligned}
&(a \circledcirc b)\mathrm{vcod} \wedge_v (c \circledcirc d)\mathrm{vdom} \\
&= \Big((a|_* a\mathrm{hcod} \wedge_h b\mathrm{hdom}) \circ (a\mathrm{hcod} \wedge_h b\mathrm{hdom}_*|b)\Big)\mathrm{vcod} \\
&\qquad \wedge_v \Big((c|_* c\mathrm{hcod} \wedge_h d\mathrm{hdom}) \circ (c\mathrm{hcod} \wedge_h d\mathrm{hdom}_*|d)\Big)\mathrm{vdom} \\
&= \Big((a|_* a\mathrm{hcod} \wedge_h b\mathrm{hdom})\mathrm{vcod} \circ (a\mathrm{hcod} \wedge_h b\mathrm{hdom}_*|b)\mathrm{vcod}\Big) \\
&\qquad \wedge_v \Big((c|_* c\mathrm{hcod} \wedge_h d\mathrm{hdom})\mathrm{vdom} \circ (c\mathrm{hcod} \wedge_h d\mathrm{hdom}_*|d)\mathrm{vdom}\Big) \\
&=_{(iv)} \Big((a|_* a\mathrm{hcod} \wedge_h b\mathrm{hdom})\mathrm{vcod} \wedge_v (c|_* c\mathrm{hcod} \wedge_h d\mathrm{hdom})\mathrm{vdom}\Big) \\
&\qquad \circ \Big((a\mathrm{hcod} \wedge_h b\mathrm{hdom}_*|b)\mathrm{vcod} \wedge_v (c\mathrm{hcod} \wedge_h d\mathrm{hdom}_*|d)\mathrm{vdom}\Big)
\end{aligned}$$

Plugging this result into Equation 2, we get

$$\begin{aligned}
&=_3 \Big[(a|_* a\mathrm{hcod} \wedge_h b\mathrm{hdom}) \circ (a\mathrm{hcod} \wedge_h b\mathrm{hdom}_*|b)\Big|_* \\
&\qquad ((a|_* a\mathrm{hcod} \wedge_h b\mathrm{hdom})\mathrm{vcod} \wedge_v (c|_* c\mathrm{hcod} \wedge_h d\mathrm{hdom})\mathrm{vdom}) \\
&\qquad\qquad \circ ((a\mathrm{hcod} \wedge_h b\mathrm{hdom}_*|b)\mathrm{vcod} \wedge_v (c\mathrm{hcod} \wedge_h d\mathrm{hdom}_*|d)\mathrm{vdom})\Big] \\
&\quad \bullet \Big[((a|_* a\mathrm{hcod} \wedge_h b\mathrm{hdom})\mathrm{vcod} \wedge_v (c|_* c\mathrm{hcod} \wedge_h d\mathrm{hdom})\mathrm{vdom}) \\
&\qquad \circ ((a\mathrm{hcod} \wedge_h b\mathrm{hdom}_*|b)\mathrm{vcod} \wedge_v (c\mathrm{hcod} \wedge_h d\mathrm{hdom}_*|d)\mathrm{vdom})_*\Big| \\
&\qquad\qquad (c|_* c\mathrm{hcod} \wedge_h d\mathrm{hdom}) \circ (c\mathrm{hcod} \wedge_h d\mathrm{hdom}_*|d)\Big]
\end{aligned}$$

As a reminder of where we are in this calculation, we provide the following picture, showing that we have established the above product as a vertical product of horizontal products (that is, horizontally (co)restrict and then vertically (co)restrict):



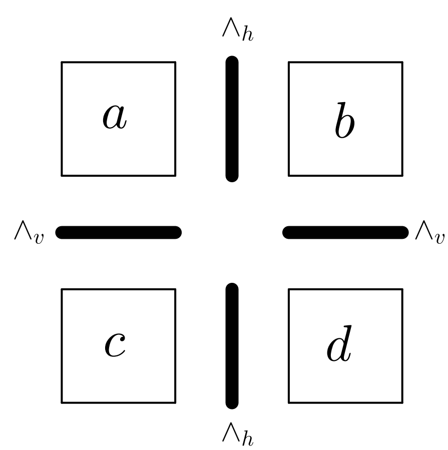
We apply Axiom (iii):

$$
\begin{aligned}
=_4 \Big\{ &\Big[(a|_* a\text{hcod} \wedge_h b\text{hdom})\Big|_* \\
&((a|_* a\text{hcod} \wedge_h b\text{hdom})\text{vcod} \wedge_v (c|_* c\text{hcod} \wedge_h d\text{hdom})\text{vdom})\Big] \\
&\circ \Big[(a\text{hcod} \wedge_h b\text{hdom}_*|b)\Big|_* \\
&((a\text{hcod} \wedge_h b\text{hdom}_*|b)\text{vcod} \wedge_v (c\text{hcod} \wedge_h d\text{hdom}_*|d)\text{vdom})\Big]\Big\} \\
\bullet \Big\{ &\Big[((a|_* a\text{hcod} \wedge_h b\text{hdom})\text{vcod} \wedge_v (c|_* c\text{hcod} \wedge_h d\text{hdom})\text{vdom})_*\Big| \\
&(c|c\text{hcod} \wedge_h d\text{hdom})\Big] \\
&\circ \Big[((a\text{hcod} \wedge_h b\text{hdom}_*|b)\text{vcod} \wedge_v (c\text{hcod} \wedge_h d\text{hdom}_*|d)\text{vdom})_*\Big| \\
&(c\text{hcod} \wedge_h d\text{hdom}_*|d)\Big]\Big\}
\end{aligned}
$$

We now apply middle-four of the double cells in $\mathcal{G}$:

$$
\begin{aligned}
=_5 \Big\{ &\Big[(a|_* a\text{hcod} \wedge_h b\text{hdom}) \Big|_* \\
&((a|_* a\text{hcod} \wedge_h b\text{hdom})\text{vcod} \wedge_v (c|_* c\text{hcod} \wedge_h d\text{hdom})\text{vdom}) \Big] \\
&\quad \bullet \Big[((a|_* a\text{hcod} \wedge_h b\text{hdom})\text{vcod} \wedge_v (c|_* c\text{hcod} \wedge_h d\text{hdom})\text{vdom})_* \Big| \\
&\qquad (c|_* c\text{hcod} \wedge_h d\text{hdom}) \Big] \Big\} \\
\circ \Big\{ &\Big[(a\text{hcod} \wedge_h b\text{hdom}_* | b) \Big|_* \\
&((a\text{hcod} \wedge_h b\text{hdom}_* | b)\text{vcod} \wedge_v (c\text{hcod} \wedge_h d\text{hdom}_* | d)\text{vdom}) \Big] \\
&\quad \bullet \Big[((a\text{hcod} \wedge_h b\text{hdom}_* | b)\text{vcod} \wedge_v (c\text{hcod} \wedge_h d\text{hdom}_* | d)\text{vdom})_* \Big| \\
&\qquad (c\text{hcod} \wedge_h d\text{hdom}_* | d) \Big] \Big\}
\end{aligned}
$$

The preceding calculations broke up the horizontal (co)restrictions in the sense that we now have four double cells instead of the original two. This allowed us to use the middle-four interchange law on the following cells:



We now "reassemble" the four double cells. Note now the following (the axiom from which the line follows is in the subscript of the equals sign):

$$\Big[(a|_*a\text{hcod} \wedge_h b\text{hdom})\Big|_*(a|_*a\text{hcod} \wedge_h b\text{hdom})\text{vcod} \wedge_v (c|_*c\text{hcod} \wedge_h d\text{hcod})\text{vdom})\Big]$$

$$=_{(ix)} \Big[(a|_*a\text{hcod} \wedge_h b\text{hdom})\Big|_*$$

$$(a\text{vcod}|_*(a\text{hcod} \wedge_h b\text{hdom})\text{vcod}) \wedge_v (c\text{vdom}|_*(c\text{hcod} \wedge_h d\text{hdom})\text{vdom})\Big]$$

$$=_{(v)} \Big[(a|_*a\text{hcod} \wedge_h b\text{hdom})\Big|_*$$

$$(a\text{vcod} \wedge_v c\text{vdom}|_*(a\text{hcod} \wedge_h b\text{hdom})\text{vcod} \wedge_v (c\text{hcod} \wedge_h d\text{hdom})\text{vdom})\Big]$$

$$=_{(viii)} \Big[(a|_*a\text{hcod} \wedge_h b\text{hdom})\Big|_*$$

$$(a\text{vcod} \wedge_v c\text{vdom}|_*$$

$$(a\text{hcodvcod} \wedge_h b\text{hdomvcod}) \wedge_v (c\text{hcodvdom} \wedge_h d\text{hdomvdom}))\Big]$$

$$=_{(vii)} \Big[(a|_*a\text{hcod} \wedge_h b\text{hdom})\Big|_*$$

$$(a\text{vcod} \wedge_v c\text{vdom}|_*$$

$$(a\text{hcodvcod} \wedge_v c\text{hcodvdom}) \wedge_h (b\text{hdomvcod} \wedge_v d\text{hdomvdom}))\Big]$$

$$=_{(viii)} \Big[(a|_*a\text{hcod} \wedge_h b\text{hdom})\Big|_*$$

$$(a\text{vcod} \wedge_v c\text{vdom}|_*(a\text{vcod} \wedge_v c\text{vdom})\text{hcod} \wedge_h (b\text{vcod} \wedge_v d\text{vdom})\text{hdom})\Big]$$

$$=_{(vi)} \Big([a|_*a\text{vcod} \wedge_v c\text{vdom}]\Big|_*$$

$$[a\text{hcod} \wedge_h b\text{hdom}|_*(a\text{vcod} \wedge_v c\text{vdom})\text{hcod} \wedge_h (b\text{vcod} \wedge_v d\text{vdom})\text{hdom}]\Big)$$

$$=_{(v)} \Big([a|_*a\text{vcod} \wedge_v c\text{vdom}]\Big|_*$$

$$[a\text{hcod}|_*(a\text{vcod} \wedge_v c\text{vdom})\text{hcod}] \wedge_h [b\text{hdom}|_*(b\text{vcod} \wedge_v d\text{vdom})\text{hdom}]\Big)$$

$$=_{(ix)} \Big([a|_*a\text{vcod} \wedge_v c\text{vdom}]\Big|_*$$

$$[a|_*a\text{vcod} \wedge_v c\text{vdom}]\text{hcod} \wedge_h [b|_*b\text{vcod} \wedge_v d\text{vdom}]\text{hdom}\Big)$$

Similarly, we have the following three facts:

$$\Big[((a|_* a\text{hcod} \wedge_h b\text{hdom})\text{vcod} \wedge_v (c|_* c\text{hcod} \wedge_h d\text{hdom})\text{vdom})_* \Big| (c|_* c\text{hcod} \wedge_h d\text{hdom})\Big]$$
$$= \Big([a\text{vcod} \wedge_v c\text{vdom}_*|c]\Big|_* [a\text{vcod} \wedge_v c\text{vdom}_*|c]\text{hcod} \wedge_h [b\text{vcod} \wedge_v d\text{vdom}_*|d]\text{hdom}\Big),$$

$$\Big[(a\text{hcod} \wedge_h b\text{hdom}_*|b)\Big|_* ((a\text{hcod} \wedge_h b\text{hdom}_*|b)\text{vcod} \wedge_v (c\text{hcod} \wedge_h d\text{hdom}_*|d)\text{vdom})\Big]$$
$$= \Big([a|_* a\text{vcod} \wedge_v c\text{vdom}]\text{hcod} \wedge_h [b|_* b\text{vcod} \wedge_v d\text{vdom}]\text{hdom}_* \Big| [b|_* b\text{vcod} \wedge_v d\text{vdom}]\Big),$$

and

$$\Big[((a\text{hcod} \wedge_h b\text{hdom}_*|b)\text{vcod} \wedge_v (c\text{hcod} \wedge_h d\text{hdom}_*|d)\text{vdom})_* \Big| (c\text{hcod} \wedge_h d\text{hdom}_*|d)\Big]$$
$$= \Big([a\text{vcod} \wedge_v c\text{vdom}_*|c]\text{hcod} \wedge_h [b\text{vcod} \wedge_v d\text{vdom}_*|d]\text{hdom}_* \Big| [b\text{vcod} \wedge_v d\text{vdom}_*|d]\Big).$$

Then equation 5 above becomes

$$\begin{aligned}
=_6 \Big\{ &\Big([a|_* a\text{vcod} \wedge_v c\text{vdom}]\Big|_* \\
&[a|_* a\text{vcod} \wedge_v c\text{vdom}]\text{hcod} \wedge_h [b|_* b\text{vcod} \wedge_v d\text{vdom}]\text{hdom}\Big) \\
&\quad \bullet \Big([a\text{vcod} \wedge_v c\text{vdom}_*|c]\Big|_* \\
&\qquad [a\text{vcod} \wedge_v c\text{vdom}_*|c]\text{hcod} \wedge_h [b\text{vcod} \wedge_v d\text{vdom}_*|d]\text{hdom}\Big)\Big\} \\
\circ \Big\{ &\Big([a|_* a\text{vcod} \wedge_v c\text{vdom}]\text{hcod} \wedge_h [b|_* b\text{vcod} \wedge_v d\text{vdom}]\text{hdom}_* \Big| \\
&[b|_* b\text{vcod} \wedge_v d\text{vdom}]\Big) \\
&\quad \bullet \Big([a\text{vcod} \wedge_v c\text{vdom}_*|c]\text{hcod} \wedge_h [b\text{vcod} \wedge_v d\text{vdom}_*|d]\text{hdom}_* \Big| \\
&\qquad [b\text{vcod} \wedge_v d\text{vdom}_*|d]\Big)\Big\}
\end{aligned}$$

We now apply Axiom (iii):

$$=_7 \Big([a|_* a\text{vcod} \wedge_v c\text{vdom}] \bullet [a\text{vcod} \wedge_v c\text{vdom}_*|c]\Big|_*$$

$$([a|_* a\text{vcod} \wedge_v c\text{vdom}]\text{hcod} \wedge_h [b|_* b\text{vcod} \wedge_v d\text{vdom}]\text{hdom})$$

$$\bullet ([a\text{vcod} \wedge_v c\text{vdom}_*|c]\text{hcod} \wedge_h [b\text{vcod} \wedge_v d\text{vdom}_*|d]\text{hdom})\Big)$$

$$\circ \Big(([a|_* a\text{vcod} \wedge_v c\text{vdom}]\text{hcod} \wedge_h [b|_* b\text{vcod} \wedge_v d\text{vdom}]\text{hdom})$$

$$\bullet ([a\text{vcod} \wedge_v c\text{vdom}_*|c]\text{hcod} \wedge_h [b\text{vcod} \wedge_v d\text{vdom}_*|d]\text{hdom})_*\Big|$$

$$[b|_* b\text{vcod} \wedge_v d\text{vdom}] \bullet [b\text{vcod} \wedge_v d\text{vdom}_*|d]\Big)$$

We now apply the definition of $\odot$:

$$=_8 \Big(a \odot c\Big|_*([a|_* a\text{vcod} \wedge_v c\text{vdom}]\text{hcod} \wedge_h [b|_* b\text{vcod} \wedge_v d\text{vdom}]\text{hdom})$$

$$\bullet ([a\text{vcod} \wedge_v c\text{vdom}_*|c]\text{hcod} \wedge_h [b\text{vcod} \wedge_v d\text{vdom}_*|d]\text{hdom})\Big)$$

$$\circ \Big(([a|_* a\text{vcod} \wedge_v c\text{vdom}]\text{hcod} \wedge_h [b|_* b\text{vcod} \wedge_v d\text{vdom}]\text{hdom})$$

$$\bullet ([a\text{vcod} \wedge_v c\text{vdom}_*|c]\text{hcod} \wedge_h [b\text{vcod} \wedge_v d\text{vdom}_*|d]\text{hdom})_*\Big|b \odot d\Big)$$

Note now that (again, the justifying axioms are in the subscripts):

$$([a|_* a\text{vcod} \wedge_v c\text{vdom}]\text{hcod} \wedge_h [b|_* b\text{vcod} \wedge_v d\text{vdom}]\text{hdom})$$

$$\bullet ([a\text{vcod} \wedge_v c\text{vdom}_*|c]\text{hcod} \wedge_h [b\text{vcod} \wedge_v d\text{vdom}_*|d]\text{hdom})$$

$$=_{(iv)} ([a|_* a\text{vcod} \wedge_v c\text{vdom}]\text{hcod} \bullet [a\text{vcod} \wedge_v c\text{vdom}_*|c]\text{hcod})$$

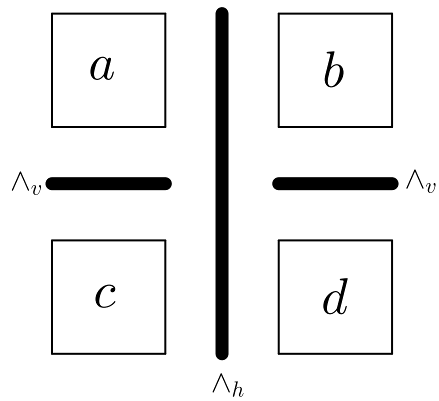
$$\wedge_h ([b|_* b\text{vcod} \wedge_v d\text{vdom}]\text{hdom} \bullet [b\text{vcod} \wedge_v d\text{vdom}_*|d]\text{hdom})$$

$$=_{(ix)} ([a|_* a\text{vcod} \wedge_v c\text{vdom}] \bullet [a\text{vcod} \wedge_v c\text{vdom}_*|c])\text{hcod}$$

$$\wedge_h ([b|_* b\text{vcod} \wedge_v d\text{vdom}] \bullet [b\text{vcod} \wedge_v d\text{vdom}_*|d])\text{hdom}$$

$$= (a \odot c)\text{hcod} \wedge_h (b \odot d)\text{hdom}$$

The preceding calculations have successfully reassembled the four double cells into the two in which we are interested. That is, we have constructed the following:



Therefore, equation 10 becomes

$$
\begin{aligned}
&=_9 \left(a \odot c\Big|_*(a \odot c)\mathrm{hcod} \wedge_h (b \odot d)\mathrm{hdom}\right) \\
&\quad \circ \left((a \odot c)\mathrm{hcod} \wedge_h (b \odot d)\mathrm{hdom}_*\Big| b \odot d\right) \\
&=_{10} (a \odot c) \circledcirc (b \odot d).
\end{aligned}
$$

Having established that the middle-four interchange law is satisfied, we have finally proved the following:

Theorem 7.2.2. *If $\mathcal{G}$ is a double inductive groupoid, then DIS$(\mathcal{G})$, as constructed in Construction 7.2.1, is a double inverse semigroup.* □

7.3 Double Inverse Semigroups to Double Inductive Groupoids

Construction 7.3.1. Given a double inverse semigroup $(S, \circledcirc, \odot)$, we construct a double inductive groupoid

$$\mathrm{DIG}(S) = (\mathrm{DIG}(S)_0, \mathrm{Ver}(\mathrm{DIG}(S)), \mathrm{Hor}(\mathrm{DIG}(S)), \mathrm{Dbl}(\mathrm{DIG}(S)))$$

as follows:

– $\mathrm{DIG}(S)_0 = E(S, \circledcirc) \cap E(S, \odot)$.

– $\mathrm{Ver}(\mathrm{DIG}(S)) = E(S, \circledcirc)$.

– $\mathrm{Hor}(\mathrm{DIG}(S)) = E(S, \odot)$.

– $\mathrm{Dbl}(\mathrm{DIG}(S)) = S(\circledcirc, \odot)$. Let a, b be any two double cells.

- We define $a\mathrm{hdom} = a \circledcirc a^{\circledcirc}$ and $a\mathrm{hcod} = a^{\circledcirc} \circledcirc a$. Whenever $a\mathrm{hcod} = b\mathrm{hdom}$, the horizontal composite is defined as $a \circ b = a \circledcirc b$. Define a horizontal partial order $\leq$ by $a \leq b$ if and only if $a = e \circ b = e \circledcirc b$ for some vertical arrow e. The horizontal meet of two vertical arrows e and f is defined to be $e \wedge_h f = e \circledcirc f$. Note that since vertical arrows are contained in $S(\circledcirc, \odot)$, this $\leq$ is also a partial order on the vertical arrows $E(S, \circledcirc)$. If we have a vertical arrow $e \leq a\mathrm{hdom}$, we define $(a|_* e) = a \circledcirc e$ and if $e \leq a\mathrm{hcod}$, we define $(e_*|a) = e \circledcirc a$.

- We define $a\mathrm{vdom} = a \odot a^{\odot}$ and $a\mathrm{vcod} = a^{\odot} \odot a$. Whenever $a\mathrm{vcod} = b\mathrm{vdom}$, the vertical composite is defined as $a \bullet b = a \odot b$. Define a vertical partial order $\lesssim$ by $a \lesssim b$ if and only if $a = e \bullet b = e \odot b$ for some horizontal arrow e. The vertical meet of two horizontal arrows e and f is defined to be $e \wedge_v f = e \odot f$. Note that since horizontal arrows are contained in $S(\circledcirc, \odot)$, this $\lesssim$ is also a partial order on the horizontal arrows $E(S, \odot)$. If we have a horizontal arrow $e \lesssim a\mathrm{vdom}$, we define $[a|_* e] = a \odot e$ and if $e \lesssim a\mathrm{vcod}$, we define $[a_*|e] = e \odot a$. ♢

Note. • It is not immediately obvious that the intersection $E(S, \odot) \cap E(S, \circledcirc)$ is non-empty. It may be the case that there are no shared idempotents. This however is *not* the case: if $a \in S$, then it follows that $(a \odot a^{\odot}) \circledcirc (a \odot a^{\odot})^{\circledcirc} \in E(S, \circledcirc)$, since $a \circledcirc a^{\circledcirc}$ is idempotent for any $a \in S$ (the product of an element with its inverse is idempotent). We also note that

$$\begin{aligned}
&[(a \odot a^{\odot}) \circledcirc (a \odot a^{\odot})^{\circledcirc}] \odot [(a \odot a^{\odot}) \circledcirc (a \odot a^{\odot})^{\circledcirc}] \\
&= [(a \odot a^{\odot}) \odot (a \odot a^{\odot})] \circledcirc [(a \odot a^{\odot})^{\odot} \circledcirc (a \odot a^{\odot})^{\circledcirc}] \\
&= (a \odot a^{\odot}) \circledcirc [(a^{\circledcirc} \odot a^{\circledcirc\odot}) \odot (a^{\circledcirc} \odot a^{\circledcirc\odot})] \\
&= (a \odot a^{\odot}) \circledcirc (a^{\circledcirc} \odot a^{\circledcirc\odot}) \\
&= (a \odot a^{\odot}) \circledcirc (a \odot a^{\odot})^{\circledcirc}.
\end{aligned}$$

Therefore, $(a \odot a^{\odot}) \circledcirc (a \odot a^{\odot})^{\circledcirc} \in E(S, \odot)$ and $E(S, \odot) \cap E(S, \circledcirc) \neq \varnothing$.

- It is known from the previous chapter (Lawson's construction) that this construction is such that

$$(\mathrm{Ver}(\mathrm{DIG}(S)), \mathrm{Dbl}(\mathrm{DIG}(S))) \text{ and } (\mathrm{Hor}(\mathrm{DIG}(S)), \mathrm{Dbl}(\mathrm{DIG}(S)))$$

  are both inductive groupoids with the orders, meets, and (co)restrictions defined as above.

- We have a groupoid structure in both directions and the horizontal and vertical compositions are defined by the horizontal and vertical semigroup products, respectively. These semigroup products satisfy middle-four and thus so do the compositions. That is, if S is a double inverse semigroup, then $\mathrm{DIG}(S)$, as constructed, is indeed a well defined double groupoid.

There will now be some lemmas showing that this construction satisfies all additional axioms of the DIGs.

Recall that any double inverse semigroup is commutative by Kock's Theorem 3.2.7; in the following proofs, there will be repeated use of the commutativity of S.

Lemma 7.3.2. *Let $S(\circledcirc, \odot)$ be a double inverse semigroup and let $a, b \in S$. Then*

(i) $(a \circledcirc b)^{\odot} = a^{\odot} \circledcirc b^{\odot}$.

(ii) $(a \odot b)^{\circledcirc} = a^{\circledcirc} \odot b^{\circledcirc}$.

Proof. We will prove only (i); that (ii) is true follows analogously. We first note that, by repeated use of middle-four,

$$\begin{aligned}(a^{\odot} \circledcirc b^{\odot}) \odot ((a \circledcirc b) \odot (a^{\odot} \circledcirc b^{\odot})) &= (a^{\odot} \circledcirc b^{\odot}) \odot ((a \odot a^{\odot}) \circledcirc (b \odot b^{\odot})) \\ &= (a^{\odot} \odot (a \odot a^{\odot})) \circledcirc (b^{\odot} \odot (b \odot b^{\odot})) \\ &= a^{\odot} \odot b^{\odot}.\end{aligned}$$

Similarly,

$$(a \circledcirc b) \odot (a^{\odot} \circledcirc b^{\odot}) \odot (a \circledcirc b) = a \circledcirc b.$$

Then, by the definition of semigroup inverse, $(a \circledcirc b)^{\odot} = a^{\odot} \circledcirc b^{\odot}$. □

Lemma 7.3.3. *Let $S(\circledcirc, \odot)$ be a double inverse semigroup and let $a, b \in S$. Then*

(i) $(a^{\odot} \odot a) \circledcirc (b^{\odot} \odot b) = (a \circledcirc b)^{\odot} \odot (a \circledcirc b)$.

(ii) $(a^{\circledcirc} \circledcirc a) \odot (b^{\circledcirc} \circledcirc b) = (a \odot b)^{\circledcirc} \circledcirc (a \odot b)$.

Proof. We prove only (i); that (ii) is true follows analogously. Now, by the preceding proposition and middle-four in S,

$$\begin{aligned}(a \circledcirc b)^{\odot} \odot (a \circledcirc b) &= (a^{\odot} \circledcirc b^{\odot}) \odot (a \circledcirc b) \\ &= (a^{\odot} \odot a) \circledcirc (b^{\odot} \odot b)\end{aligned}$$ □

Lemma 7.3.4. *Let $(S, \odot)$ be a commutative inverse semigroup and let $a, b \in S$. Then*

If $a^{\odot} \odot a = b^{\odot} \odot b$, then $(a \odot b)^{\odot} \odot (a \odot b) = a^{\odot} \odot a = b^{\odot} \odot b$.

Proof. Because each of the fact that $a^{\odot} \odot a = b^{\odot} \odot b$, the commutativity of S and the fact that $a^{\odot} \odot a$ is idempotent,

$$\begin{aligned}(a \odot b)^{\odot} \odot (a \odot b) &= (b^{\odot} \odot a^{\odot}) \odot (a \odot b) \\ &= (a^{\odot} \odot a) \odot (b^{\odot} \odot b) \\ &= (a^{\odot} \odot a) \odot (a^{\odot} \odot a) \\ &= a^{\odot} \odot a. \end{aligned}$$ □

The preceding lemmas show that composition in the constructed DIG(S) is well-defined. We now check that all the axioms of double inductive groupoids are satisfied. For the following lemmas, we will prove only the first statement, since all other statements in each lemma follow analogously.

Lemma 7.3.5. *(Axiom (iii)) If a, b are double cells, f', g' are horizontal arrows (idempotents with respect to $\odot$) and f, g are vertical arrows (idempotents with respect to $\circledcirc$ in DIG(S)), the following laws about restrictions and corestrictions preserving composition hold:*

(a) $\big(a \bullet b|_* f \bullet g\big) = \big(a|_* f\big) \bullet \big(b|_* g\big)$.

(b) $\big[a \circ b|_* f' \circ g'\big] = \big[a|_* f'\big] \circ \big[b|_* g'\big]$.

(c) $\big(f \bullet g_*|a \bullet b\big) = \big(f_*|a\big) \bullet \big(g_*|b\big)$.

(d) $\big[f' \circ g'{}_*|a \circ b\big] = \big[f'{}_*|a\big] \circ \big[g_*|b\big]$.

Proof. By the commutativity of $\odot$ and middle-four in S, we have that

$$\begin{aligned}(a \bullet b|_* f \bullet g) &= (a \odot b) \circledcirc (f \odot) \\ &= (a \circledcirc f) \odot (b \circledcirc g) \\ &= (a|_* f) \bullet (b|_* g) \end{aligned}$$ □

Lemma 7.3.6. *(Axiom (iv)) If e, f, g and h are horizontal arrows (idempotents with respect to $\odot$) and e', f', g' and h' are vertical arrows (idempotents with respect to $\circledcirc$) in in DIG(S), the following laws about composition preserving meets hold:*

(a) $(e \wedge_v f) \circ (g \wedge_v h) = (e \circ g) \wedge_v (f \circ h)$.

(b) $(e' \wedge_h f') \bullet (g' \wedge_h h') = (e' \bullet g') \wedge_h (f' \bullet h')$.

Proof. By definition of $\wedge_v$ and $\circ$ along with middle-four in S,

$$\begin{aligned}(e \wedge_v f) \circ (g \wedge_v h) &= (e \odot f) \circledcirc (g \odot h) \\ &= (e \circledcirc g) \odot (f \circledcirc h) \\ &= (e \circ g) \wedge_v (f \circ h).\end{aligned}$$

□

Corollary 7.3.7. *(Axiom (vii)) If each of e, f, g and h are objects (idempotents with respect to both $\odot$ and $\circledcirc$) in DIG(S), the following law about commuting meets holds:*

$$(e \wedge_h f) \wedge_v (g \wedge_h h) = (e \wedge_v g) \wedge_h (f \wedge_v h).$$

Proof. Note that $(e \wedge_h f) \wedge_v (g \wedge_h h) = (e \wedge_h f) \odot (g \wedge_h h) = (e \wedge_h f) \bullet (g \wedge_h h)$ and $e(e \wedge_v g) \wedge_h (f \wedge_v h) = (e \wedge_v g) \circledcirc (f \wedge_v h) = (e \wedge_v g) \circ (f \wedge_v h)$ and apply the preceding lemma. □

Lemma 7.3.8. *(Axiom (v)) If e and g are horizontal arrows (idempotents with respect to $\odot$), e' and g' are vertical arrows (idempotents with respect to $\circledcirc$) in DIG(S) and each of f, h, f' and h' are objects (idempotents with respect to both $\odot$ and $\circledcirc$), the following laws about restrictions and corestrictions preserving meets hold:*

(a) $(e|_* f) \wedge_v (g|_* h) = (e \wedge_v g|_* f \wedge_v h)$.

(b) $[e'|_* f'] \wedge_h [g'|_* h'] = [e' \wedge_h g'|_* f' \wedge_h h']$.

(c) $(e_*|f) \wedge_v (g_*|h) = (e \wedge_v g_*|f \wedge_v h)$.

(d) $[e'_*|f'] \wedge_h [g'_*|h'] = [e' \wedge_h g'_*|f' \wedge_h h']$.

Proof. By the definition of horizontal restrictions and vertical meet, along with the fact that S satisfies middle-four:

$$\begin{aligned}(e|_* f) \wedge_v (g|_* h) &= (e \circledcirc f) \odot (g \circledcirc h)\\ &= (e \odot g) \circledcirc (f \odot h)\\ &= (e \wedge_v g|_* f \wedge_v h).\end{aligned}$$ □

Lemma 7.3.9. *(Axiom (vi)) If a is a double cell, f a horizontal arrow (idempotent with respect to $\odot$), g a vertical arrow (idempotent with respect to $\circledcirc$) and h an object in DIG(S) such that $h \leq f\text{hdom} \wedge g\text{vdom}$ (recall the image from Axiom (vi)), the following law about commuting restrictions and corestrictions holds:*

$$([a|_* f]|_* [g|_* h]) = [(a|_* g)|_* (f|_* h)]$$

Similarly quantified, here are all laws about commuting restrictions and corestrictions that must hold:

(a) $([a|_* f]|_* [g|_* h]) = [(a|_* g)|_* (f|_* h)]$.

(b) $[(a|_* g)|_* (f|_* h)] = ([a|_* f]|_* [g|_* h])$.

(c) $([h_*|g]_*|[f_*|a]) = [(h_*|f)_*|(g_*|a)]$.

(d) $[(h_*|f)_*|(g_*|a)] = ([h_*|g]_*|[f_*|a])$.

Proof. By the definition of corestrictions and by the middle-four in S, we have that

$$\begin{aligned}([a|_* f]|_* [g|_* h]) &= ([a|_* f]) \circledcirc ([g|_* h])\\ &= (a \odot f) \circledcirc (g \odot h)\\ &= (a \circledcirc g) \odot (f \circledcirc h)\\ &= (a|_* g) \odot (f|_* h)\\ &= [(a|_* g)|_* (f|_* h)].\end{aligned}$$ □

Lemma 7.3.10. *(Axiom (viii)) If e, f are vertical arrows (idempotents with respect to $\circledcirc$) and e', f' are horizontal arrows (idempotents with respect to $\odot$) in DIG(S), the following laws about domains and codomains preserving meets hold:*

(a) $(e \wedge_h f)\text{vdom} = e\text{vdom} \wedge_h f\text{vdom}$.

(b) $(e \wedge_h f)\text{vcod} = e\text{vcod} \wedge_h f\text{vcod}$.

(c) $(e' \wedge_v f')\text{hdom} = e'\text{hdom} \wedge_v f'\text{hdom}$.

(d) $(e' \wedge_v f')\text{hcod} = e'\text{hcod} \wedge_v f'\text{hcod}$.

Proof. By the definition of horizontal meet and vertical codomain together with middle-four in S,

$$\begin{aligned}
(e \wedge_h f)\text{vdom} &= (e \circledcirc f)\text{vdom} \\
&= (e \circledcirc f) \odot (e \circledcirc f)^{\odot} \\
&= (e \circledcirc f) \odot (e^{\odot} \circledcirc f^{\odot}) \\
&= (e \odot e^{\odot}) \circledcirc (f \odot f^{\odot}) \\
&= e\text{vdom} \circledcirc f\text{vdom} \\
&= e\text{vdom} \wedge_h f\text{vdom}.
\end{aligned}$$

□

Lemma 7.3.11. *(Axiom (ix)) If a is a double cell, e is a vertical arrow (idempotent with respect to $\circledcirc$) and e' is a horizontal arrow (idempotent with respect to $\odot$) in DIG(S), then the following laws about domains and codomains preserving restrictions and corestrictions hold:*

(a) $(a|_* e)\text{vdom} = (a\text{vdom}|_* e\text{vdom})$.

(b) $(a|_* e)\text{vcod} = (a\text{vcod}|_* e\text{vcod})$.

(c) $(e_*|a)\text{vdom} = (e\text{vdom}_*|a\text{vdom})$.

(d) $(e_*|a)\text{vcod} = (e\text{vcod}_*|a\text{vcod})$.

(e) $[a|_*e']\text{hdom} = [a\text{hdom}|_*e'\text{hdom}]$.

(f) $[a|_*e']\text{hcod} = [a\text{hcod}|_*e'\text{hcod}]$.

(g) $[e'{}_*|a]\text{hdom} = [e'\text{vdom}_*|a\text{hdom}]$.

(h) $[e'{}_*|a]\text{hcod} = [e'\text{hcod}_*|a\text{hcod}]$.

Proof. By the definition of vertical domain, middle-four in S and the definition of horizontal restriction,

$$\begin{aligned}(a|_*e)\text{vdom} &= (a \circledcirc e)\text{vdom} \\ &= (a \circledcirc e) \odot (a \circledcirc e)^{\odot} \\ &= (a \circledcirc e) \odot (a^{\odot} \circledcirc e^{\odot}) \\ &= (a \odot a^{\odot}) \circledcirc (e \odot e^{\odot}) \\ &= a\text{vdom} \circledcirc e\text{vdom} \\ &= (a\text{vdom}|_*e\text{vdom}).\end{aligned}$$

□

Having proved all of the preceding lemmas, we have proved the following theorem:

Theorem 7.3.12. *If $S(\circledcirc, \odot)$ is a double inverse semigroup, then $DIG(S)$, as constructed in Construction 7.3.1, is a double inductive groupoid.*

7.4 An Isomorphism of Categories

In the previous chapter, we constructed an isomorphism of categories between the category of inductive groupoids with inductive functors and the category of inverse semigroups with semigroup homomorphisms. The goal of this section is to establish an analogous result; we will find an isomorphism of categories between the category of double inductive groupoids with double inductive functors and the category of double inverse semigroups and double semigroup homomorphisms. We first make the following two natural definitions:

Definition 7.4.1. Let $\mathcal{G}$ and $\mathcal{G}'$ be double inductive groupoids. A *double inductive functor* $f : \mathcal{G} \to \mathcal{G}'$ is a double functor whose vertical arrow, horizontal arrow and double cell functions preserve all partial orders and meets. ■

Definition 7.4.2. Let $(S, \circledcirc, \odot)$ and $(S', \circledcirc', \odot')$ be double inverse semigroups. A *double semigroup homomorphism* $\varphi : S \to S'$ is a function $\varphi : S \to S'$ such that, for all $a, b \in S$, $(a \odot b)\varphi = a\varphi \odot b\varphi$ and $(a \circledcirc b)\varphi = a\varphi \circledcirc b\varphi$. ■

We will make use of a simple, yet important, observation which follows directly from the definition of double semigroup homomorphism.

Note. Let $\varphi : (S, \circledcirc, \odot) \to (S', \circledcirc', \odot')$ be a double semigroup homomorphism. If $e \in E(S, \odot)$ is an idempotent with respect to $\odot$, then $e\varphi = (e \odot e)\varphi = e\varphi \odot' e\varphi$ and $e\varphi \in E(S', \odot')$ and thus $E(S, \odot) \subseteq E(S, \odot)\varphi$. Similarly, $E(S, \circledcirc) \subseteq E(S, \circledcirc)\varphi$. This tells us that φ preserves idempotents in both directions.

We will also make some notational choice for ease of discussion:

Notation. We denote the category of double inductive groupoids with double inductive functors as **DIG** and we denote the category of double inverse semigroups with double semigroup homomorphisms as **DIS**.

We may now state and prove the following theorem:

Theorem 7.4.3. *There exists an isomorphism of categories between* **DIG** *and* **DIS**.

Proof. We define a pair of functors

$$\mathbf{DIG} \underset{F'}{\overset{F}{\rightleftarrows}} \mathbf{DIS}$$

as follows:

(i) $F : \mathbf{DIG} \to \mathbf{DIS}$:

- On objects: For any double inductive groupoid $\mathcal{G}$, define $\mathcal{G}F = \text{DIS}(\mathcal{G})$, as defined in Construction 7.2.1. Recall that $\text{DIS}(\mathcal{G}) = \text{Dbl}(\mathcal{G})$ with products defined, for any $a, b \in \text{DIS}(\mathcal{G})$, as

$$a \circledcirc b = (a|_* a\text{hcod} \wedge_h b\text{hdom}) \circ (a\text{hcod} \wedge_h b\text{hdom}_*|b)$$
$$a \odot b = [a|_* a\text{vcod} \wedge_v b\text{vdom}] \bullet [a\text{vcod} \wedge_v b\text{vdom}_*|b]$$

- On arrows: For any double inductive functor $f : \mathcal{G} \to \mathcal{G}'$, define fF : $\text{DIS}(\mathcal{G}) \to \text{DIS}(\mathcal{G}')$ to be the double cell function f_d of f.

Since the arrow function of F returns the double cell function of a double functors, functoriality is immediate from the definition of functor composition. One also notes that, for any $a, b \in \text{DIS}(\mathcal{G})$, we have

$$\begin{aligned}
(a \odot b)fF &= (a \odot b)f_d \\
&= \Big([a|_* a\text{vcod} \wedge_v b\text{vdom}] \bullet [a\text{vcod} \wedge_v b\text{vdom}_*|b]\Big) f_d \\
&= \Big([a|_* a\text{vcod} \wedge_v b\text{vdom}]\Big) f_d \bullet' \Big([a\text{vcod} \wedge_v b\text{vdom}_*|b]\Big) f_d \\
&\qquad (f_d \text{ preserves composition}) \\
&= [af_d|_*(a\text{vcod} \wedge_v b\text{vdom})f_d] \bullet' [(a\text{vcod} \wedge_v b\text{vdom})f_{d*}|bf_d] \\
&\qquad (f_d \text{ preserves (co)restrictions}) \\
&= [af_d|_* a\text{vcod}f_d \wedge_v b\text{vdom}f_d] \bullet' [a\text{vcod}f_d \wedge_v b\text{vdom}f_{d*}|bf_d] \\
&\qquad (f_d \text{ preserves meets}) \\
&= [af_d|_* af_d\text{vcod} \wedge_v bf_d\text{vdom}] \bullet' [af_d\text{vcod} \wedge_v bf_d\text{vdom}_*|bf_d] \\
&\qquad (f_d \text{ preserves (co)domains}) \\
&= af_d \odot' bf_d \\
&= afF \odot' bfF.
\end{aligned}$$

Similarly, we have that $(a \circledcirc b)fF = afF \circledcirc' bfF$. That is, fF is justifiably a double semigroup homomorphism.

(ii) $F' : \mathbf{DIS} \to \mathbf{DIG}$:

- On objects: For any double inverse semigroup S, define $SF' = \mathrm{DIG}(S)$, as defined in Construction 7.3.1. Recall that $\mathrm{DIG}(S)$ has the following data:
  - $\mathrm{DIG}(S)_0 = E(S, \odot) \cap E(S, \circledcirc)$.
  - $\mathrm{Ver}(\mathrm{DIG}(S)) = E(S, \circledcirc)$.
  - $\mathrm{Hor}(\mathrm{DIG}(S)) = E(S, \odot)$.
  - $\mathrm{Dbl}(\mathrm{DIG}(S)) = S(\odot, \circledcirc)$.
- On arrows: For any double semigroup homomorphism $\varphi : S \to S'$ between double inverse semigroups, define $\varphi F' : \mathrm{DIG}(S) \to \mathrm{DIG}(S')$ to be the double (inductive) functor with the following data:
  - An object function defined to be φ restricted by $E(S, \odot) \cap E(S, \circledcirc)$.
  - A vertical arrow function defined to be φ restricted by $E(S, \circledcirc)$.
  - A horizontal arrow function defined to be φ restricted by $E(S, \odot)$.
  - A double cell function defined to be φ.

The above defined object, vertical arrow and horizontal arrow functions are well-defined due to the fact that double semigroup homomorphisms preserve idempotents, as made clear in the note at the beginning of this section. As with the preceding functor, the functoriality of F' follows immediately from the the definition of functor composition. The work is in showing that $\varphi F'$ is actually inductive (since $\varphi F'$ is either φ or a restriction of φ, we will write only φ) :

(a) We check that φ preserves all partial orders. If $a, b \in \mathrm{DIG}(S)$ with $a \leq b$, then by definition $a = e \circ b = e \circledcirc b$ for some $e \in \mathrm{Ver}(DIS) = E(S, \circledcirc)$. Since

φ is a homomorphism, then,

$$
\begin{aligned}
& a = e \circledcirc b \\
\Longrightarrow \quad & a\varphi = (e \circledcirc b)\varphi \\
\Longrightarrow \quad & a\varphi = e\varphi \circledcirc' b\varphi \\
\Longrightarrow \quad & a\varphi = e\varphi \circ' b\varphi
\end{aligned}
$$

Since φ preserves idempotents, $e\varphi$ is indeed a vertical arrow and thus $a\varphi \leq' b\varphi$ in $DIG(S')$. Similarly, if $a \lesssim b$ in DIG(S), then $a\varphi \lesssim' b\varphi$ in $DIG(S')$.

(b) Since φ preserves the orders, φ preserves all meets.

(c) Since φ preserves all orders and all meets, φ preserves all (co)restrictions.

It can then be said that the functor defined in the arrow function of F' is justifiably inductive.

We now check that these two functors compose to the identity functors. We will first make a note about the arrow functions:

If f is any inductive functor, then fF is the double cell function and thus fFF' is a functor whose double cell function is indeed just the the double cell function of f. The object, vertical arrow and horizontal arrow functions of fFF' are also just the object, vertical arrow and horizontal functions of f. For example, the vertical arrow function of fFF' is the restriction of fF to the idempotents of the horizontal operation in the given double inverse semigroup. However, these idempotents are exactly the vertical arrows and thus the restriction of fF by the horizontal idempotents is exactly the vertical arrow function of f.

If φ is any double semigroup homomorphism, then the double cell function of $\varphi F'$ is just φ. Then $\varphi F' F = \varphi$, since $\varphi F' F$ is defined to be the double cell function of $\varphi F'$.

Having checked that the arrow functions compose to the identity, we check the object functions:

We know by our construction that the elements of a double inverse semigroup S are exactly the double cells of DIG(S) and that the double cells of a double inductive groupoid $\mathcal{G}$ are exactly the elements of DIS($\mathcal{G}$). Then it is the case that the elements of DIS(DIG(S)) are exactly the elements of S and the double cells of DIG(DIS($\mathcal{G}$)) are exactly the double cells of $\mathcal{G}$.

We show that the products of elements in DIS(DIG(S)) are the same as those in S. If $a, b \in S$, we consider the product $a \odot b$. In DIS(DIG(S)), this product is

$$\begin{aligned}
&[a|_* a\text{vcod} \wedge_v b\text{vdom}] \bullet [a\text{vcod} \wedge_v b\text{vdom}] \\
&= a \odot (a^{\odot} \odot a) \odot (b \odot b^{\odot}) \odot (a^{\odot} \odot a) \odot (b \odot b^{\odot}) \odot b \\
&= a \odot ((a^{\odot} \odot a) \odot (a^{\odot} \odot a)) \odot ((b \odot b^{\odot}) \odot (b \odot b^{\odot})) \odot b \\
&= (a \odot a^{\odot} \odot a) \odot (b \odot b^{\odot} \odot b) \\
&= a \odot b.
\end{aligned}$$

Similarly, $a \circledcirc b$ in DIS(DIG(S)) is the same as in S. Since the elements and the products are the same, we can say that DIS(DIG(S)) $= S$.

It will finally be shown that the composites of double cells in $\mathcal{G}FF' =$ DIG(DIS($\mathcal{G}$)) are the same as those in $\mathcal{G}$. We will then be done since vertical and horizontal arrows can be considered as identity double cells for horizontal and vertical composition, respectively. For any double cells $a, b \in \mathcal{G}$, if the composite $a \bullet b$ exists, we know that

$$a \odot b = [a|_* a\text{vcod} \wedge_v b\text{vdom}] \bullet [a\text{vcod} \wedge_v b\text{vdom}_* |b]$$

in the double inverse semigroup DIS($\mathcal{G}$). However, since the composite $a \bullet b$ exists, $a$vcod $= b$vdom and so this product in DIG(DIS($\mathcal{G}$)), then, becomes

$$\begin{aligned}
[a|_* a\text{vcod}] \bullet [b\text{vdom}_* |b] &= (a \odot a^{\odot} \odot a) \bullet (b \odot b^{\odot} \odot b) \\
&= a \bullet b.
\end{aligned}$$

Similarly, $a \circ b$ in DIG(DIS($\mathcal{G}$)) is the same as in $\mathcal{G}$ and we are done. Since the double cells (and thus horizontal and vertical arrows) and the composites are the same, we can say that DIG(DIS($\mathcal{G}$)) $= \mathcal{G}$. □

7.5 Special Properties of Double Inverse Semigroups and Double Inductive Groupoids

Naturally, after having established an isomorphism of categories between **DIS** and **DIG**, one would like to use this isomorphism as a means of formulating a characterisation of double inverse semigroups.

Lemma 7.5.1. *Let $(S, \odot, \circledcirc)$ be a double inverse semigroup. For all $a, b \in E(S, \odot) \cap E(S, \circledcirc)$,*

$$a \odot b = a \circledcirc b.$$

Proof. We first note that, since a and b are idempotent with respect to $\odot$ and $\circledcirc$,

$$a = a \odot a = a \circledcirc a$$

and

$$b = b \odot b = b \circledcirc b.$$

Also,

$$\begin{aligned}(a \circledcirc b) \odot (a \circledcirc b) &= (a \odot a) \circledcirc (b \odot b)\\ &= a \circledcirc b.\end{aligned}$$

Using these facts and the commutativity in a double inverse semigroup,

$$
\begin{aligned}
a \odot b &= (a \odot b) \circledcirc (a \odot b) \\
&= (a \odot b) \circledcirc (b \odot a) \\
&= (a \circledcirc b) \odot (b \circledcirc a) \\
&= (a \circledcirc b) \odot (a \circledcirc b) \\
&= a \circledcirc b.
\end{aligned}
$$

□

Our isomorphism allows us to think of the partial orders, meets and (co)restrictions as semigroup products. Because, on the objects, these semigroup products coincide, we have the following corollary:

Corollary 7.5.2. *Let $\mathcal{G}$ be a double inductive groupoid. For all objects $a, b \in \mathrm{Obj}(\mathcal{G})$,*

$$a \leq b \text{ if and only if } a \lesssim b$$

and

$$a \wedge_h b = a \wedge_v b.$$

□

Note. We can consider both the set of vertical (horizontal) arrows and objects in a double inductive groupoid as posetal categories, whose objects are the vertical (horizontal) arrows or objects of the groupoid, respectively, and there is a unique arrow $u \to v$ if and only if $u \leq v$ ($u \lesssim v$). In the following discussions, whenever we write $\mathrm{Obj}(\mathcal{G})$, $\mathrm{Ver}(\mathcal{G})$ or $\mathrm{Hor}(\mathcal{G})$, we are considering them as posetal categories.

Having established that the two order structures of a double inductive groupoid coincide on its objects, we seek to further study and describe the relationship between the order on the objects and the orders on both the horizontal and vertical arrows.

Let $\mathcal{G}$ be a double inductive groupoid and let S be its corresponding double inverse semigroup. Given a double cell (element of S) a, it will have the following form:

$$\begin{array}{ccc} A & \xrightarrow{a \odot a^{\odot}} & B \\ {\scriptstyle a \circledcirc a^{\circledcirc}}\downarrow & a & \downarrow{\scriptstyle a^{\circledcirc} \circledcirc a} \\ C & \xrightarrow[a^{\odot} \odot a]{} & D \end{array}$$

For convenience, let $a_h = a \odot a^{\odot}$, $a_v = a \circledcirc a^{\circledcirc}$. By commutativity, a now looks like

$$\begin{array}{ccc} A & \xrightarrow{a_h} & B \\ {\scriptstyle a_v}\downarrow\!\bullet & a & \bullet\!\downarrow{\scriptstyle a_v} \\ C & \xrightarrow[a_h]{} & D \end{array}$$

Note that

$$A = a_h\text{hdom} = a_h\text{hcod} = B = a_v\text{vdom} = a_v\text{vcod} = D = a_h\text{vcod} = C.$$

Then, for every double cell $a \in \text{Dbl}(\mathcal{G})$, a has the following form:

$$\begin{array}{ccc} A & \xrightarrow{a_h} & A \\ {\scriptstyle a_v}\downarrow\!\bullet & a & \bullet\!\downarrow{\scriptstyle a_v} \\ A & \xrightarrow[a_h]{} & A \end{array}$$

Let $\mathcal{G}$ be a double inductive groupoid and let A be an object of $\mathcal{G}$. Then there is a natural collection of double cells

$$(A)\mathcal{S} = \left\{ a \in \text{Dbl}(\mathcal{G}) \;\middle|\; \begin{array}{ccc} A & \xrightarrow{a_h} & A \\ {\scriptstyle a_v}\downarrow\!\bullet & a & \bullet\!\downarrow{\scriptstyle a_v} \\ A & \xrightarrow[a_h]{} & A \end{array} \right\}$$

Though definitely a double groupoid, it is not immediately obvious that $A\mathcal{S}$ is a double *inductive* groupoid. That is, it could be possible that meets or (co)restrictions may not be well defined on $A\mathcal{S}$ (the meet of two arrows in $A\mathcal{S}$ may not be in $A\mathcal{S}$, for example). The following proposition, however, allows us to properly call these one-object double inductive groupoids:

Proposition 7.5.3. *For each object* $A \in \mathrm{Obj}(\mathcal{G})$*, the above-defined collection* $A\mathcal{S}$ *is a one-object double inductive groupoid.*

Proof. These objects are subobjects of double groupoids are thus double groupoids themselves. We now check the following properties:

(i) Vertical meets of horizontal arrows in $A\mathcal{S}$ are again horizontal arrows in $A\mathcal{S}$.

(ii) Horizontal meets of vertical arrows in $A\mathcal{S}$ are again vertical arrows in $A\mathcal{S}$.

(iii) Horizontal (co)restrictions of double cells in $A\mathcal{S}$ by vertical arrows in $A\mathcal{S}$ are again double cells of $A\mathcal{S}$.

(iv) Vertical (co)restrictions of double cells in $A\mathcal{S}$ by horizontal arrows in $A\mathcal{S}$ are again double cells of $A\mathcal{S}$.

To prove (i), let f and g be horizontal arrows in $A\mathcal{S}$. We must show that the horizontal domain and codomain of $f \wedge_v g$ are both A. This is true since, in $\mathcal{G}$, we have the property that meets preserve domains and codomains. That is,

$$(f \wedge_v g)\mathrm{hdom} = f\mathrm{hdom} \wedge_v g\mathrm{hdom} = A \wedge_v A = A.$$

Similarly, $(f \wedge_v g)\mathrm{hcod} = A$.

The proof of (ii) is similar to that of (i).

To prove (iii), we note that if we have a cell $a \in A\mathcal{S}$ and a vertical arrow $e \leq a\mathrm{hdom}$ in $A\mathcal{S}$, then the domain of the restriction $(e_* | a)$ is e. Since $e \in A\mathcal{S}$, the domain and codomain of e are both A. Because all four corners of a double cell are equal, all four corners of $(e_* | a)$ has all four corners A and thus lives inside of $A\mathcal{S}$.

The proof of (iv) is similar to that of (iii).

These one-object double groupoids, then, contain all of its meets and (co)restrictions. Because they are subobjects of a double inductive groupoid, they must satisfy all the axioms of a double inductive groupoid. Therefore, these one-object double groupoids are indeed one-object double inductive groupoids. □

By our isomorphism of categories, we can consider these one-object double inductive groupoids as a special class of double inverse semigroups whose operations share only one idempotent. The following theorem provides two ways of characterising these one-object double inductive groupoids:

Theorem 7.5.4. *If A is a single-object double inductive groupoid, then the following statements are equivalent:*

(a) A is an Abelian group.

(b) A has only one vertical and horizontal arrow.

(c) The two natural partial order relations on the cells of A coincide.

Proof. $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$: If A is an Abelian group, we can consider A as a double inverse semigroup whose operations are both the group operation of A. Group operations are inverse semigroup operations which have exactly one idempotent. Each operation, then, has one idempotent and they are both the same. That is, there is only one vertical and horizontal arrow: this idempotent. Since the inverse operations of A are the same, we have that $a \leq b$ if and only if $a = e_A \circledcirc b = e_A \odot b$ if and only if $a \lesssim b$ and thus the order relations must coincide on the cells (elements) A.
$(b) \Rightarrow (a)$: If there is only one vertical and horizontal arrow, every double cell is composable in either direction. More specifically, each of the horizontal/vertical restrictions are trivial and thus the inverse semigroup operations reduce to the compositions in the double inductive groupoid associated with A. The compositions, however are group operations and A is therefore an Abelian group by Eckmann-Hilton.
$(c) \Rightarrow (b)$: Obviously, since the order relations coincide, so do the two meets. Note that if u and v are vertical arrows, then they are horizontal idempotents. Also, since we are in a single-object double inductive groupoid, all restrictions and corestrictions of vertical arrows are trivial and the semigroup products when restricted to the

vertical arrows are simply the groupoid compositions. If $u \leq v$, then

$$
\begin{aligned}
& u = u \circledcirc v \\
\Rightarrow \quad & u \circledcirc u = u \circledcirc v \\
\Rightarrow \quad & u \circ u = u \circ v \\
\Rightarrow \quad & u = v.
\end{aligned}
$$

The last step follows from cancellation in a groupoid. We can then conclude that there is only one vertical arrow, call it u. Similarly, there is only one horizontal arrow, call it f. □

The preceding theorem becomes provides an especially useful characterisation. This becomes obvious with the following:

Proposition 7.5.5. *A single-object double inductive groupoid A has only one vertical arrow and one horizontal arrow.*

Proof. We first recall that in any inductive groupoid, horizontal composition of horizontal arrows preserves the vertical meets. That is,

$$(f \circ g) \wedge_v (f' \circ g') = (f \wedge_v f') \circ (g \wedge_v g').$$

We note that $a \wedge_v b = a$ implies that $a \lesssim b$ and thus preservation of meets in this way implies the following law about preserving the vertical partial order:

$$f \lesssim f', g \lesssim g' \text{ implies } f \circ g \lesssim f' \circ g'$$

Of course, in a single-object double inductive groupoid, all arrows have the same domain and codomain and are thus guaranteed to be composable.

We now show that horizontal inverses preserve the vertical partial order. Let $f \lesssim g$ be horizontal arrows. We can use our isomorphism of categories to consider A as a double inverse semigroup, so that $g = e \odot f$ for some horizontal arrow e. Recall that inverses of one operation distribute over the other and thus $(e^{\circledcirc})\odot(e^{\circledcirc}) = (e\odot e)^{\circledcirc} = e^{\circledcirc}$ (horizontal arrows are idempotents with respect to the vertical operation). Then $e^{\circledcirc}$ is a horizontal arrow and therefore, when we take the horizontal inverses of f and g :

$$g^{\circledcirc} = (e \odot f)^{\circledcirc} = e^{\circledcirc} \odot f^{\circledcirc} \lesssim f^{\circledcirc}.$$

Since the horizontal arrows form a meet-semilattice, there must be a minimal element with respect to $\lesssim$, call it x. That is, if f is a horizontal arrow, then $x \lesssim f$. Then by the above discussion, $x^{-1} \lesssim f^{-1}$. Trivially, $f \lesssim f$ and $x^{-1} \lesssim x^{-1}$ and thus by preservation of composition,

$$x^{-1} \circ f \lesssim f^{-1} \circ f = \mathrm{id}$$

and

$$\mathrm{id} = x^{-1} \circ x \lesssim x^{-1} \circ f.$$

That is, the only horizontal arrow is id. Similarly, there is only the vertical identity arrow, 1, and this proposition is proved. □

We can then apply the preceding two theorems to prove:

Theorem 7.5.6. *Single-object double inductive groupoids are exactly Abelian groups.*

□

If $A \in \mathrm{Obj}(\mathcal{G})$ and $a \in (A)\mathcal{S}$ and $e \leq A$ is an object (i.e., is both a horizontal and vertical arrow), then we know that the unique restriction is in $(e)\mathcal{S}$ and has the form

$$\begin{CD} e @>{e\circledcirc a_h}>> e \\ @V{e\circledcirc a_v}VV e\circledcirc a @VV{e\circledcirc a_v}V \\ e @>>{e\circledcirc a_h}> e \end{CD}$$

We now consider the following map between $(A)\mathcal{S}$ and $(e)\mathcal{S}$:

$$\varphi : (A)\mathcal{S} \to (e)\mathcal{S}$$
$$a \mapsto e \circledcirc a$$

If $a, b \in (A)\mathcal{S}$, then

$$\begin{aligned}(a \odot b)\varphi &= e \circledcirc (a \odot b) \\ &= (e \odot e) \circledcirc (a \odot b) \\ &= (e \circledcirc a) \odot (e \circledcirc b) \\ &= (a)\varphi \odot (b)\varphi.\end{aligned}$$

That is, $\varphi : (A)\mathcal{S} \to (e)\mathcal{S}$ is an Abelian group homomorphism.

Definition 7.5.7. If $\mathbf{C}$ and $\mathbf{V}$ are categories, a $\mathbf{V}-$*valued presheaf of* $\mathbf{C}$ is a contravariant functor

$$F : \mathbf{C}^{\text{op}} \to \mathbf{V}.$$ ■

The above discussion gives us a $\mathbf{Ab}-$valued presheaf

$$\mathcal{S} : \text{Obj}(\mathcal{G})^{\text{op}} \to \mathbf{Ab}.$$

On objects we can send A to $(A)\mathcal{S}$ and send an arrow $A \leq B$ to the Abelian group homomorphism induced (as described above) between $(B)\mathcal{S}$ and $(A)\mathcal{S}$.

The discussion in this section provides a construction proving the following result:

Proposition 7.5.8. *Arbitrary double inverse semigroups can be made into a* **Ab**-*valued presheaf over a meet-semilattice.*

A stronger and more desirable result, however, would be to establish an isomorphism of categories between the category of double inductive groupoids and presheaves of Abelian groups over meet-semilattices. We do that now. We first, however, introduce the notion of a morphism between two presheaves of Abelian groups on meet-semilattices:

Definition 7.5.9. A morphism of presheaves of Abelian groups on meet-semilattices $P : L^{\mathrm{op}} \to \mathbf{Ab}$ and $P' : (L')^{\mathrm{op}} \to \mathbf{Ab}$ is an ordered pair

$$(f, \{\varphi_A\}_{A \in L}) : P \to P'$$

consisting of an order and meet preserving function $f : L \to L'$ and a family of group homomorphisms $\{\varphi_A\}$ indexed by the objects of A such that, for any objects $A \leq B$ in L, the following diagram commutes:

$$\begin{array}{ccc} AP & \longleftarrow & BP \\ {\scriptstyle \varphi_A}\downarrow & & \downarrow{\scriptstyle \varphi_B} \\ (Af)P' & \longleftarrow & (Bf)P' \end{array}$$

■

To ease the following proof, we make a notational remark:

Notation. We denote the category of presheaves of Abelian groups on meet-semilattices with presheaf morphisms by **AbPSMS**.

Finally, we may prove the following:

Theorem 7.5.10. *The categories* **AbPSMS** *and* **DIG** *(and thus* **DIS**) *are isomorphic.*

Proof. We define a pair of functors

$$\mathbf{DIG} \underset{F'}{\overset{F}{\rightleftarrows}} \mathbf{AbPSMS}$$

as follows:

(i) $F : \mathbf{DIG} \to \mathbf{AbPSMS}$:

- On objects: If $\mathcal{G}$ is a double inductive groupoid, define $\mathcal{G}F$ to be the presheaf P of Abelian groups on the meet-semilattice $\text{Obj}(\mathcal{G})$ as detailed above (i.e., given by the restrictions).
- On arrows: Given a double inductive functor $f : \mathcal{G} \to \mathcal{G}'$, define a morphism of presheaves $(g, \{\varphi_A\}_{A \in \text{Obj}(\mathcal{G})})$ with
  - $g = f_0$, the object function of f.
  - $\varphi_A = f_d|_{AP}$, the double cell function of f restricted to those cells who have all corners A.

This is indeed functorial, since the double cell function of f preserves both composition and identities. This is also indeed an arrow in the AbPSMS. First, f_0 preserves all meets and orders and thus g does. For any objects $A \leq B$ in $\text{Obj}(\mathcal{G})$, we know that there is a unique Abelian group homomorphism $BP \to AP$ and thus the required diagram commutes.

(ii) $F' : \mathbf{AbPSMS} \to \mathbf{DIG}$:

- On objects: If $P : L^{\text{op}} \to \mathbf{Ab}$ is a presheaf of Abelian groups on a meet-semilattice, define a double inductive groupoid $\mathcal{G} = PF'$ with the following data:
  - Objects: $\text{Obj}(\mathcal{G}) = L$
  - Vertical/horizontal arrows: $\text{Ver}(\mathcal{G}) = \text{Hor}(\mathcal{G} = \{e_{AP} : A \in L\}$ (the identities of the Abelian groups in the image of P).
  - Double cells: $\text{Dbl}(\mathcal{G}) = \coprod_{A \in L} AP$, the disjoint union of all Abelian groups in the image of P.
- On arrows: If $(f, \{\varphi_A\}_{A \in L}) : P \to P'$ is a morphism of presheaves, define a double inductive functor $g = (f, \{\varphi_A\}_{A \in L})F' : PF' \to P'F'$ with the following data:
  - An object function: $g_0 = f$.

– A vertical/horizontal arrow function: $g_v = g_h = g_d|_{\mathrm{Ver=Hor}}$, the double cell function restricted to the vertical and horizontal arrows (the group identities).

– A double cell function: $g_d = \varphi$ defined by $g_d(a) = \varphi_{a\mathrm{hdomvdom}}(a)$ (evaluate each a using the group homomorphism whose index is the object in the four corners of a).

This is functorial, since it is composed of group homomorphisms. That is, composition (the group products) and identities are preserved. Also,

(a) This preserves all partial orders, since f is an order preserving map.

(b) Similarly, f preserves meets, so this functor does, too.

(c) Since F' preserves all orders and all meets, F' preserves all (co)restrictions.

It can then be said that the functor defined in the arrow function of F' is justifiably inductive.

We now check that these two functors compose to the identity functors.

Object functions: Given a presheaf $P : L^{\mathrm{op}} \to \mathbf{Ab}$, we check that $PF'F = P$. First, PF' is the double inductive groupoid with objects L, vertical/horizontal arrows the identity morphisms and double cells the disjoint union of the Abelian groups AP for $A \in L$. When we send this inductive groupoid into presheaves, we have a presheaf $P' : L^{\mathrm{op}} \to \mathbf{Ab}$. Note that P and P' are presheaves of Abelian groups on the same meet-semilattice. We know the two presheaf structures are given by the restrictions in the corresponding one-object double inductive groupoids. These restrictions, however, are unique and thus $PF'F = P' = P$. Conversely, suppose that we are given a double inductive groupoid $\mathcal{G}$. We check that $\mathcal{G}FF' = \mathcal{G}$. We know that $\mathcal{G}F$ is a presheaf $P : \mathrm{Obj}(\mathcal{G})^{\mathrm{op}} \to \mathbf{Ab}$. Sending this presheaf into double inductive groupoids, then, gives us a double inductive groupoid with objects $\mathrm{Obj}(\mathcal{G})$, vertical/horizontal arrows all identities and double cell the disjoint union of the groups in the image of P. It was shown before, however, that a double inductive groupoid consists solely of cells that lie inside of these groups and have only identities for vertical and horizontal arrows. It is clear, then, that this is the same double inductive groupoid, or that $\mathcal{G}FF' = \mathcal{G}$.

Arrow Functions: The arrow functions compose almost trivially to the identity, since each double functor is defined from a presheaf morphism using isomorphisms (equality), and vice-versa, and knowing that the object functions compose to the identity, ensuring that the structures we are moving through are the same.

Having defined two functors whose composition is the identity functor in either way, we have completed the proof. □

We have shown that double inverse semigroups are exactly presheaves of Abelian groups on meet-semilattices. In particular, we have seen that double inverse semigroups consist of a collection of idempotents on which the horizontal and vertical operations coincide. Add to this Kock's result that double inverse semigroups and we have the following result:

Theorem 7.5.11. *Double inverse semigroups are exactly commutative inverse semigroups.* □

The following example constructs a double inverse semigroup from a chain of Abelian groups. This example provides a good example of how the above properties of double inductive groupoids together with the isomorphisms described can be applied to construct double inverse semigroups.

Example 7.5.12. Let

$$A_0 = \mathbb{Z}_1, A_1 = \mathbb{Z}_2, \ldots, A_{n-1} = \mathbb{Z}_{2^{n-1}}, A_n = \mathbb{Z}_{2^n}$$

be a sequence of Abelian groups with identities $e_0, e_1, \ldots, e_n$ such that e_i is the identity of A_i for $i \in \{0, 1, \ldots, n\}$. For any $0 \leq m \leq n$ (as integers), note that $A_m = \mathbb{Z}_{2^m}$. For any $0 \leq m \leq n$, we denote the generator of A_m as a_m. We recall that each Abelian group can be represented as

$$A_m = (e_m)\mathcal{S} = \left\{ \begin{array}{ccc} e_m & \xrightarrow{aa^{-1}=e_m} & e_m \\ {\scriptstyle e_m}\big\downarrow & a & \big\downarrow{\scriptstyle e_m} \\ e_m & \xrightarrow[e_m]{} & e_m \end{array} \;\middle|\; a \in A_m = \mathbb{Z}_{2^m} \right\}$$

Define a partial order on the identities above by $e_1 \leq e_2 \leq \ldots \leq e_n$. We show there is a presheaf structure on this sequence of Abelian groups over this meet-semilattice. That is, $e_m \leq e_{m'}$ implies the existence of an Abelian group homomorphism $\varphi_{m',m} : A_{m'} \to A_m$. We note that $e_m \leq e_{m'}$ implies that $m \leq m'$ (as integers). Then there is therefore a well defined quotient map

$$\begin{aligned} \varphi_{m',m} : \mathbb{Z}_{2^{m'}} &\longrightarrow \mathbb{Z}_{2^m} \\ a_{m'}^j &\longmapsto a_m^{j \pmod{2^m}} \end{aligned}$$

and, since the composite of any of these φ maps is again a φ map, there is indeed a presheaf structure of this sequence of Abelian groups defined on the meet-semilattice of identities.

We can then use our isomorphism to make a double inductive groupoid $\mathcal{G}$ with the following data:

– Objects: $\mathrm{Obj}(\mathcal{G}) = \{e_0, e_1, \ldots, e_n\}$.

– Arrows: $\mathrm{Ver}(\mathcal{G}) = \mathrm{Hor}(\mathcal{G}) = \{e_0, e_1, \ldots, e_n\}$.

– Double cells: $\mathrm{Dbl}(\mathcal{G}) = \coprod_{i=0}^n A_i$, the disjoint union of the underlying sets of the A_i.

Let $a \in \mathrm{Dbl}(\mathcal{G})$. Then $a \in \coprod_{i=0}^n A_i$ and thus $a \in A_m$ for some $0 \leq m \leq n$ and has the form

$$\begin{array}{ccc} e_m & \xrightarrow{aa^{-1}=e_m} & e_m \\ {\scriptstyle e_m}\big\downarrow & a & \big\downarrow{\scriptstyle e_m} \\ e_m & \xrightarrow[e_m]{} & e_m \end{array}$$

as detailed above. It then makes sense to represent each $a \in \mathrm{Dbl}(\mathcal{G})$ as an ordered pair (a, e_m), which will tell us which Abelian group in the disjoint union a belongs to. Horizontal composition of double cells (a, e_m) and (b, e_j) is defined whenever $a\text{hcod} = e_m = e_j = a\text{hdom}$ and is thus the product in the Abelian group $A_m = A_j$. Vertical composition is defined in the same way. Since all idempotents are shared ones, it must be that the compositions coincide.

If (a, e_m) is a double cell of $\mathcal{G}$ and $e_j \leq e_m = a\text{hdom}$, then by the existence of horizontal restrictions, there is a unique double cell $((e_{j*}|a), e_j) \leq (a, e_m)$. We will do only horizontal since the horizontal/vertical operations on the shared idempotents – all of them – coincide and therefore the horizontal/vertical orders and restrictions do, too. Consider the quotient homomorphism $\varphi_{m,j} : A_m \to A_j$. Then $(a, e_m)\varphi_{m,j} = (a\varphi_{m,j}, e_j)$. The restrictions, then, are given by $((e_{j*}|a), e_j) = (a\varphi_{m,j}, e_j)$. We can write $a = a_m^k$ as some power of the generator a_m of A_m. We then can write this restriction more specifically as $(a_j^{k \,(\text{mod}\, 2^j)}, e_j)$.

We may now define a double inverse semigroup, S. Let $S = \coprod_{i=0}^{n} A_i$ and $(a, e_m), (b, e_j) \in S$, be double cells such that $e_j \leq e_m$. Write $a = a_m^k$ and $b = a_j^\ell$ as powers of generators. Then the meet $e_m \wedge_h e_j = e_j$. We also note that the homomorphism $\varphi_{j,j} : A_j \to A_j$ is the identity homomorphism, for any $0 \leq j \leq n$. Define both products (since the horizontal/vertical restrictions and operations coincide as seen above) as

$$\begin{aligned} a \odot b = a \circledcirc b &= ((a|_* e_m \wedge_h e_j), e_m \wedge_h e_j) \circ ((e_m \wedge_h e_{j*}|b), e_m \wedge_h e_j) \\ &= ((a|_* e_j), e_j) \circ ((e_{j*}|b), e_j) \\ &= (a\varphi_{m,j} \circ b\varphi_{j,j}, e_j \circ e_j) \\ &= (a_m^k \varphi_{m,j} \circ a_j^\ell \varphi_{j,j}, e_j) \\ &= (a_j^{k \,(\text{mod}\, 2^j)} \circ a_j^\ell, e_j) \\ &= (a_j^{\ell + k \,(\text{mod}\, 2^j)}, e_j) \end{aligned}$$

where $\circ$ is the operation in the group A_j. ▲

Chapter 8

Conclusion

This thesis first introduced the reader to the notion of double semigroups. We then explored some known results about the commutativity in double semigroups. In particular, we saw that all double inverse semigroups are commutative. We then introduced the notion of a double category and showed that a known construction of inverse semigroups with zero from categories, when generalised to constructing double inverse semigroups from double categories, does not provide a non-trivial correspondence. The notion of inductive groupoids is introduced and a known construction of inverse semigroups from inductive groupoids, and vice-versa, is explored.

In an effort to generalise this construction to double inverse semigroups, we defined double inductive groupoids and defined constructions of double inverse semigroups from these double inductive groupoids, and vice-versa. We showed that these constructions are functorial in nature and admit an isomorphism of categories between the category of double inductive groupoids with double inductive functors and the category of double inverse semigroups and double semigroup homomorphisms. Using this isomorphism, we defined a special class of single-object double inductive groupoids, which are ultimately Abelian groups, and showed that any double inverse semigroup can be seen as a presheaf of Abelian groups on a meet semi-lattice, namely the shared idempotents of its two operations. Using these correspondences, we concluded that double inverse semigroups are exactly commutative inverse semigroups.

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