Using Series to Construct Pythagorean Triples

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Abstract. This article provides a series-focused approach to computing Pythagorean triples.

Introduction

Geometric and arithmetic sequences and finite series are typically introduced in secondary school. Though sequences as ordered lists lend themselves to many intuitive and manageable applications, it is often difficult for students to put series, even finite, into context (González-Martín et al., 2011).

It is well known that learning mathematics is greatly enhanced and facilitated by being exposed to multiple perspectives of the same topic (Liljedahl et al., 2016). This is achieved primarily through developing problem-solving skills and techniques, which are then applied to novel contexts (Liljedahl et al., 2016). It is therefore desirable to have concrete and interesting examples of applying series, which include varied mathematical concepts and familiar contexts to encourage cross-pollination. Another benefit of repeated exposure to series is the improvement of one's comfort level ahead of seeing them as a common concept in calculus, algebra, and geometry contexts (Jones, 2011).

The authors, while discussing interesting applications of finite series, developed a construction which makes finite series concrete through computing *Pythagorean triples*; a Pythagorean triple is a triple (x, y, z) of natural numbers satisfying the Pythagorean equation $x^2 + y^2 = z^2$. The Pythagorean equation first arises geometrically as relating the length of the hypotenuse z of a right triangle with its two other sides of length x and y. Computing Pythagorean triples then has applications in measuring, for example, the length of an angled roof given its height and depth. In navigation, the Pythagorean equation gives the shortest distance z between two points $(x, 0)$ and $(0, y)$. Related to this, the Pythagorean equation is also the equation of a circle centered at

$(0,0)$ with radius z. Most commonly, one computes Pythagorean triples algebraically by solving $z = \sqrt{x^2 + y^2}$ or using trigonometric relationships. Algebraic and trigonometric techniques naturally lean heavily on irrational numbers since they rely on the use of the square root function and familiarity with trigonometric functions. Irrational numbers are not intuitive, and they are hard to compute without using a calculator. Our construction illuminates the beauty of Pythagorean triples using only natural numbers while reinforcing the utility and applicability of finite series.

Our construction relies primarily on the following elementary fact: for each natural number r, r^2 is the sum of the first r odd numbers (e.g., $5^2 = 25 = 1 + 3 + 5 + 7 + 9$ is the sum of the first 5 odd numbers), which can be expressed as the finite series

$$r^2 = \sum_{n=1}^{r} (2n - 1).$$

If one has not seen this series representation of a square as a sum of odd numbers, it is an interesting example that provides additional exposition to applications of arithmetic sequences. It is crucial that one has a good grasp of this series since it will be used to compute Pythagorean triples by solving the equivalent equation $x^2 = z^2 - y^2$, writing $z^2 - y^2$ as a difference of finite series. One can develop an intuition for this series geometrically. For example, Figure 1 shows the creation of a square with sides of length $r = 5$. The square has area $r^2 = 5^2 = 25$ and is created by taking 1 square, then successively adding on L-shaped pieces with 3, 5, 7, and 9 squares. The resulting square has area $r^2 = 25 = 1 + 3 + 5 + 7 + 9$, the sum of the first $r = 5$ odd integers.

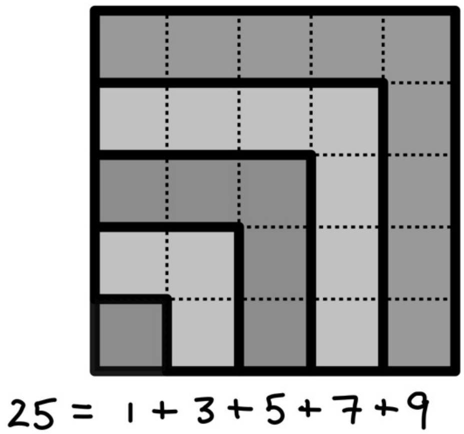

Figure 1: The Sum of the First 5 Odd Integers as Layers of L-Shaped Figures

In this paper, we will give a novel construction of Pythagorean triples. We then provide examples of using our construction to construct Pythagorean triples, which includes a discussion of when it fails to produce a Pythagorean triple. Finally, we will motivate the development of our construction and prove that it works as stated. In addition to the algebraic proof that our construction works, a geometric demonstration of the proof is provided.

The Construction

We provide a construction for the Pythagorean triple $(x, y, y + \ell)$ for any given natural number ℓ. This is accomplished by looking at every natural number x greater than ℓ and performing a check on those x values. If an x value passes our check, then we can use it to define a y value that fits into the Pythagorean triple. If x fails our check, then no Pythagorean is generated for those values of x and ℓ, and we move on to the next possible x value.

Construction.

1. Choose a natural number ℓ.

2. Choose $x > \ell$ such that x^2/ℓ and ℓ are either both even or both odd. This step can be sped up by checking only those x and ℓ that are themselves either both even or both odd. Note that we need to only check those x such that x^2/ℓ is natural since only natural numbers can be even or odd.

3. Define $y = \frac{1}{2}(x^2/\ell - \ell)$.

4. Finally, let $z = y + \ell$.

In the next section, we will show how to use this construction to create a Pythagorean triple and when it fails to do so. We also summarize the results of this construction in Table 1 for several values of ℓ and x. After gaining some intuition for this construction through examples and detailing the motivation of this construction, we will prove that this construction always yields a Pythagorean triple for appropriate values of x.

Examples

We demonstrate our construction successfully producing a Pythagorean triple. If $\ell = 2$ and $x = 4$, we have $x^2/\ell = 16/2 = 8$ which is also even. Therefore, we set $y = \frac{1}{2}(x^2/\ell - \ell) = \frac{1}{2}(8 - 2) = 3$. With $z = y + \ell = 3 + 2 = 5$, we have produced the Pythagorean triple $(4,3,5)$, or $(3,4,5)$. As we will see, our construction was inspired by the difference of two series of odd numbers; we see that $x^2 = 5^2 - 3^2$ is the sum of the two odd numbers from the 4th odd number to the 5th odd number, or $7 + 9 = 16 = 4^2$.

Of course, our construction can fail to produce a Pythagorean triple. The construction will fail to produce a Pythagorean triple most obviously when x^2/ℓ is non-natural. For example, if $\ell = 3$ and $x = 5$, then $x^2/\ell \notin \mathbb{N}$ and we cannot use our construction to produce such a Pythagorean triple. In fact, the proof that our construction works will show that $x^2/\ell \in \mathbb{N}$ is a necessary condition for x and ℓ to be part of a Pythagorean triple of the form $(x, y, y + \ell)$.

The construction also fails to produce a Pythagorean triple when ℓ and $\frac{x^2}{\ell}$ are not both even or both odd. For example, if $\ell = 4$ and $x = 10$, then ℓ is even and $\frac{x^2}{\ell}$ is odd. This shows that ℓ and x^2/ℓ can differ in being even or odd, though x and ℓ are themselves both even. Again, the proof that our construction works will show that ℓ and $\frac{x^2}{\ell}$ being both even or both odd is necessary since y is half of their difference. Otherwise, their difference would be odd and y would be non-natural.

Table 1 summarizes the outputs of our construction for various values of ℓ and x. Priority is given to primitive triples to illustrate the usefulness of this construction. Generating multiples of previously generated triples or their primitive factors is avoided. We encourage the reader to create their own Pythagorean triples by choosing some $\ell > 4$ following the construction for several different x. One will notice that a lot of repeated or multiples of previously generated triples will arise. An interested reader may explore the question of novelty; that is, how common are repeats and how rare are new triples?

Table 1: The output and intermediate values of our construction.

$\boldsymbol{\ell}$	$\boldsymbol{x}$	$\boldsymbol{x^2/\ell}$	**Are $\boldsymbol{\ell}$ and $\boldsymbol{x^2/\ell}$ both even or both odd?**	$\boldsymbol{y = \frac{1}{2}(x^2/\ell - \ell)}$	$\boldsymbol{z = y + \ell}$	**Pythagorean Triple**
2	4	8	Yes	3	5	(4,3,5)
2	8	32	Yes	15	17	(8,15,17)
3	5	$\frac{25}{3} \notin \mathbb{N}$	N/A	N/A	N/A	N/A
3	15	75	Yes	36	39	(15,36,39)
3	21	147	Yes	72	75	(21,72,75)
4	6	9	No	N/A	N/A	N/A
4	10	25	No	N/A	N/A	N/A
4	24	144	Yes	70	74	(24,70,74)

The Explanation and Proof

To compute Pythagorean triples, we consider the equivalent equation $x^2 = z^2 - y^2$ and write x^2 as the sum of ℓ consecutive odd integers using the finite series representation of the squares z^2 and y^2. If $z = y + 1$, for example, we see that x^2 is a single odd integer ($\ell = 1$), namely the $(y+1)$th, or the zth, odd number:

$$\begin{aligned} x^2 &= (y+1)^2 - y^2 \\ &= \sum_{n=1}^{y+1}(2n-1) - \sum_{m=1}^{y}(2m-1) \\ &= \big(1 + 3 + \cdots + (2y-1) + (2(y+1)-1)\big) - \big(1 + 3 + \cdots + (2y-1)\big) \\ &= 2(y+1) - 1 \\ &= 2z - 1. \end{aligned}$$

To demonstrate this, consider the Pythagorean triple $(3,4,5)$:

$$\begin{aligned} 3^2 &= 5^2 - 4^2 \\ &= \sum_{n=1}^{5}(2n-1) - \sum_{m=1}^{4}(2m-1) \\ &= (1+3+5+7+9) - (1+3+5+7) \\ &= 9. \end{aligned}$$

Remark. Fibonacci (1225) presented a similar construction which agrees with ours in the case when $\ell = 1$. Fibonacci's construction says to choose a natural number y such that $2y + 1$ is a perfect square, call it x^2. Then x and y are part of a Pythagorean triple since $x^2 + y^2 = (2y+1) + y^2 = y^2 + 2y + 1 = (y+1)^2$ is also a perfect square and the Pythagorean triple is of the form $(x, y, y+1)$. In our notation, this is of the form $(x, y, y+\ell)$ with $\ell = 1$. Indeed, this is exactly the triple produced with our construction if $\ell = 1$, since $y = \frac{1}{2}(x^2/\ell\ - \ell) = \frac{1}{2}(x^2/1\ - 1) = \frac{1}{2}(x^2 - 1)$, which simplifies to $2y = x^2 - 1$, or $x^2 = 2y + 1$ as Fibonacci required. Our construction agrees with Fibonacci's, though we approach the problem from opposite directions: Fibonacci started with a suitable y and produced a corresponding x; we start with a suitable x and produce a corresponding y. Using this perspective, our construction allows us to consider values $\ell > 1$, too.

That x^2 can be written as the sum of $\ell = 1$ odd integer when $z = y + 1$ is no coincidence. Indeed, this approach can be generalized further to the setting when $z = y + \ell$ for any natural $\ell \in \mathbb{N}$. The computations for arbitrary ℓ are somewhat abstract in nature and readers may find them difficult to follow. To aid understanding, we first provide a concrete example.

Example 2. Consider the Pythagorean triple $(8,15,17) = (8,15,15 + 2)$ with $\ell = 2$. Then,

$$\begin{aligned} z^2 \quad &= 17^2 = \sum_{n=1}^{17} (2n - 1) \\ &= 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 + 27 + 29 + 31 + 33 \end{aligned}$$

is the sum of the first 17 odd numbers and

$$\begin{aligned} y^2 \quad &= 15^2 = \sum_{n=1}^{15} (2n - 1) \\ &= 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 + 27 + 29 \end{aligned}$$

is the sum of the first 15 odd numbers. Our calculation then implies that

$$x^2 = z^2 - y^2 = 17^2 - 15^2 = \sum_{n=1}^{17} (2n - 1) - \sum_{n=1}^{15} (2n - 1) = 31 + 33 = 64 = 8^2;$$

$x^2 = 17^2 - 15^2$ is the sum of the $\ell = 2$ odd numbers starting at the 16th odd number and ending at the 17th odd number: $x^2 = 31 + 33$.

With this example to help guide our understanding, we suppose that ℓ is any natural number. In this case, we have

$$\begin{aligned}
x^2 &= z^2 - y^2 \\
&= \sum_{n=1}^{z}(2n-1) - \sum_{n=1}^{y}(2n-1) \\
&= \sum_{n=1}^{y+\ell}(2n-1) - \sum_{n=1}^{y}(2n-1) \quad (\text{since } z = y + \ell) \\
&= \overbrace{\left[\sum_{n=y+1}^{y+\ell}(2n-1) + \sum_{n=1}^{y}(2n-1)\right]}^{\text{break up the sum from 1 to } y+\ell \text{ into two pieces}} - \sum_{n=1}^{y}(2n-1) \\
&= \sum_{n=y+1}^{y+\ell}(2n-1) \quad \left(\text{cancel the zero pair of } \sum_{n=1}^{y}(2n-1) \text{ terms}\right) \\
&= [2(y+1)-1] + [2(y+2)-1] + \cdots + [2(y+\ell)-1]
\end{aligned}$$

Therefore, our problem reduces to finding $\ell = z - y$ consecutive odd numbers which sum to x^2. We note that x^2/ℓ is a natural number since ℓ is necessarily a factor of x^2. This can be shown as below:

$$\begin{aligned}
x^2 &= [2(y+1)-1] + [2(y+2)-1] + \cdots + [2(y+\ell)-1] \\
&= [2y+1] + [2y+3] + \cdots + [2y+(2\ell-1)] \\
&= [\underbrace{2y + 2y + \cdots + 2y}_{\ell \text{ copies}}] + [1 + 3 + \cdots + (2\ell-1)] \\
&= \ell \cdot 2y + \sum_{n=1}^{\ell}(2n-1) \\
&= \ell \cdot 2y + \ell^2 \\
&= \ell(2y+\ell)
\end{aligned}$$

Thus we can write $\frac{x^2}{\ell} = \frac{\ell(2y+\ell)}{\ell} = 2y + \ell$, implying that ℓ and $\frac{x^2}{\ell}$ are either both even or both odd since their difference $\frac{x^2}{\ell} - \ell = 2y$ is even. This further implies that x^2 and ℓ^2, and thus x

and ℓ, are either both even or both odd since their difference

$$x^2 - \ell^2 = (\ell \cdot 2y + \ell^2) - \ell^2 = \ell \cdot 2y = 2(y\ell)$$

is always even. Though x and ℓ being both even or both odd is a necessary condition for $\frac{x^2}{\ell}$ and ℓ to be either both even or both odd, we warn the reader that this is not a sufficient condition. For example, when $\ell = 4$ and $x = 6$, Table 1 shows that this does not result in a Pythagorean triple.

Construction. Following this discussion, we restate and prove our construction:

1. Choose a natural number ℓ.

2. Choose $x > \ell$ such that x^2/ℓ and ℓ are either both even or both odd. This step can be sped up by checking only those x and ℓ that are themselves either both even or both odd. Note that we need to only check those x such that x^2/ℓ is natural since only natural numbers can be even or odd.

3. Define $y = \frac{1}{2}(x^2/\ell - \ell)$.

4. Finally, let $z = y + \ell$.

Claim. *This construction yields a Pythagorean triple.*

Proof. If x is chosen such that x^2/ℓ and ℓ are either both even or both odd, the difference $x^2/\ell - \ell$ is even and $y = \frac{1}{2}(x^2/\ell - \ell)$ is a natural number. Finally, letting $z = y + \ell$, we have

$$\begin{aligned} z^2 - y^2 &= (y + \ell)^2 - y^2 \\ &= y^2 + 2y\ell + \ell^2 - y^2 \\ &= 2y\ell + \ell^2 \\ &= 2 \cdot \frac{1}{2}(x^2/\ell - \ell)\ell + \ell^2 \\ &= (x^2 - \ell^2) + \ell^2 \\ &= x^2. \end{aligned}$$

That is, $z^2 - y^2 = x^2$, which can be re-arranged to $x^2 + y^2 = z^2$, the Pythagorean Theorem.

$\square$

We end this paper with a brief discussion on how to understand this construction geometrically. Recall that Figure 1 shows how $r^2 = 25$ is the sum of the first $r = 5$ odd integers by layering on L-shaped figures of increasing size. Figure 2(a) shows how this is generalized to a square of side y and how this square can be extended to a square of side $z = y + \ell$ by adding a sequence of ℓ one-block-thick L-shaped figures. For demonstration, we have chosen $y = 3$ and $\ell = 3$.

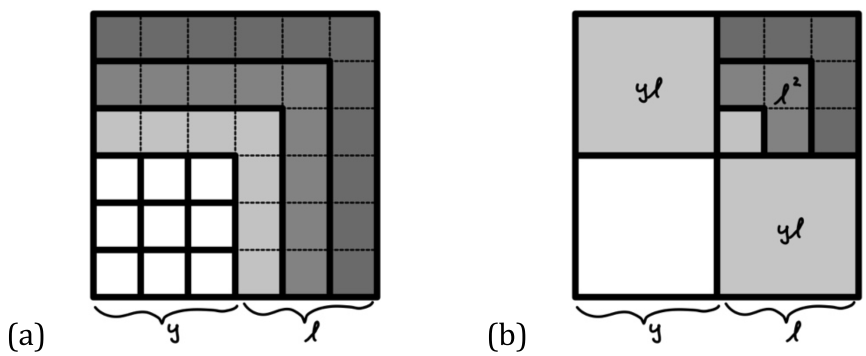

Figure 2: Graphically Representing Series of Odd Integers by Creating Squares

Figure 2(b) shows that the area added to go from y^2 to $(y + \ell)^2$ comprises two rectangles and a square. The two rectangles each have area $y\ell$ and the square ℓ^2. It is instructive to see that the square of area ℓ^2 is made up of pieces of $1, 3, \ldots, 2\ell - 1$. This is a more abstract application of the odd-squares series that motivated geometrically in Figure 1. Hence, the total area added is $2y\ell + \ell^2$ and we have $(2y\ell + \ell^2) + y^2 = (y + \ell)^2$. By the Pythagorean Theorem, it follows that $x^2 = 2y\ell + \ell^2$. One can then pick values of y and ℓ and determine the value of x that satisfies $x^2 = 2y\ell + \ell^2$. This approach yields all solutions $(x, y, y + \ell)$ where y and ℓ are natural numbers and x is either a natural or irrational number (Dickson, 1894). However, to determine whether a number is a perfect square is a tedious process, especially for larger values of x^2. It is possible to approach the problem from the other side and start instead with x. By determining the relationship between x and ℓ, that $x^2 = \ell(\ell + 2y)$, our construction's requirements are immediate: (i) $x > \ell$, (ii) x and ℓ are either both even or both odd, and (iii) x^2/ℓ and ℓ are both even or both odd. Choosing x and ℓ that meet these criteria yields y that produces a Pythagorean triple $(x, y, y + \ell)$ where x and ℓ are natural numbers and y is a rational number (DeWolf & Viswanathan, 2022). It is easier to determine if $x^2 - \ell^2$ can be divided by ℓ than it is to determine if $2y\ell + \ell^2$ is a perfect square.

Conclusions

Pythagorean triples and the Pythagorean Theorem play a natural role in applications across science, engineering, and mathematics. For example, a Pythagorean triple gives the discrete distance between two integral points in space and can be used to find the magnitude of vectors which often correspond to velocities in physics. Given that distances and magnitudes are also inherently geometric, it is not surprising that the generation of Pythagorean triples can be geometrically motivated. In this paper, we made this connection explicit by using a finite series that simultaneously represents both the geometric and algebraic perspectives. Resulting is a novel construction motivated by the finite series representation of a square r^2 as the sum of the first r odd natural numbers. This perspective proves to be fruitful in the sense that it generalizes Fibonacci's previously known construction of Pythagorean triples and illuminates the relationship between abstract mathematical concepts and real-world applications.

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