Double Groups and Semigroups

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Double Groups

Definition

A *double group* $(G, \circledcirc, \odot)$ is a set G equipped with two group operations $\odot$ and $\circledcirc$ that satisfy the the *middle-four interchange law:* for all $a, b, c, d \in S$,

$$(a \circledcirc b) \odot (c \circledcirc d) = (a \odot c) \circledcirc (b \odot d).$$

Observation 1:

$(G, \circledcirc, \odot)$ a double group.

Let $1_\odot$ be the identity for $\odot$ and $1_\circledcirc$ the identity for $\circledcirc$.

$$\begin{aligned} 1_\odot &= 1_\odot \odot 1_\odot \\ &= (1_\odot \circledcirc 1_\circledcirc) \odot (1_\circledcirc \circledcirc 1_\odot) \\ &= (1_\odot \odot 1_\circledcirc) \circledcirc (1_\circledcirc \odot 1_\odot) \\ &= 1_\circledcirc \circledcirc 1_\circledcirc \\ &= 1_\circledcirc \end{aligned}$$

Observation 1: The identities of a double group must agree.

Observation 2:

$(G, \circledcirc, \odot)$ a double group.

Let 1 be the (shared) identity for $\odot$ and $\circledcirc$.

$$\begin{aligned} a \odot b &= (a \circledcirc 1) \odot (1 \circledcirc b) \\ &= (a \odot 1) \circledcirc (1 \odot b) \\ &= a \circledcirc b \end{aligned}$$

Observation 2: The operations of a double group must agree.

Observation 3:

$(G, \circledcirc, \odot)$ a double group.

Let 1 be the (shared) identity for $\odot$ and $\circledcirc$ and write products by concatenation.

$$\begin{aligned} ab &= (1a)(b1) \\ &= (1b)(a1) \\ &= ba \end{aligned}$$

Observation 3: The operations of a double group must agree and must be commutative.

Eckmann-Hilton Argument: Double groups are essentially Abelian groups.

Double Semigroups

Definition

A *double semigroup* $(S, \circledcirc, \odot)$ is a set equipped with two associative binary operations satisfying the *middle-four interchange law*: for all $a, b, c, d \in S$,

$$(a \circledcirc b) \odot (c \circledcirc d) = (a \odot c) \circledcirc (b \odot d).$$

- Horizontal product: $a \circledcirc b = \begin{array}{|c|c|}\hline a & b \\ \hline\end{array}$.
- Vertical product: $a \odot b = \begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array}$.
- Middle-four:

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$$

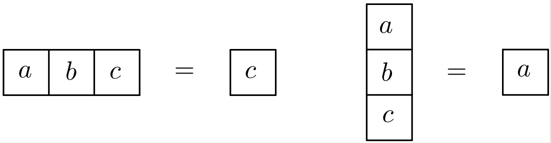
Example

Any set D can be made into a double semigroup by equipping it with left and right projection:

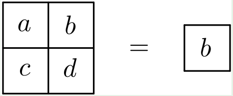
$$a \odot b = a$$

$$a \circledcirc b = b.$$

Associative:



Middle-four interchange law:



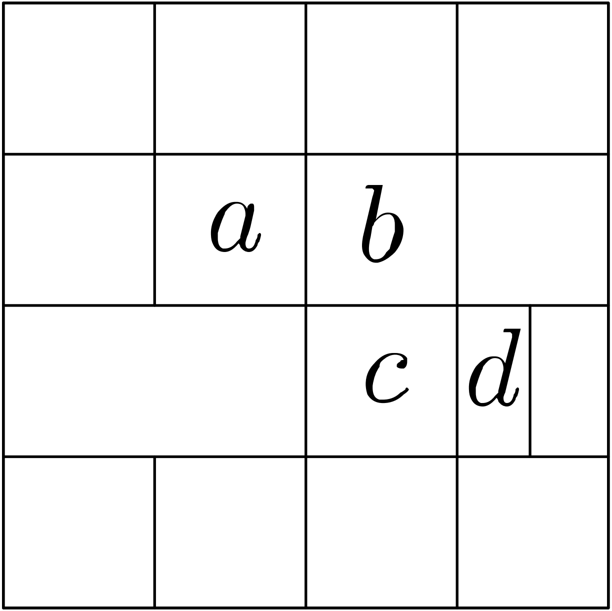
Theorem

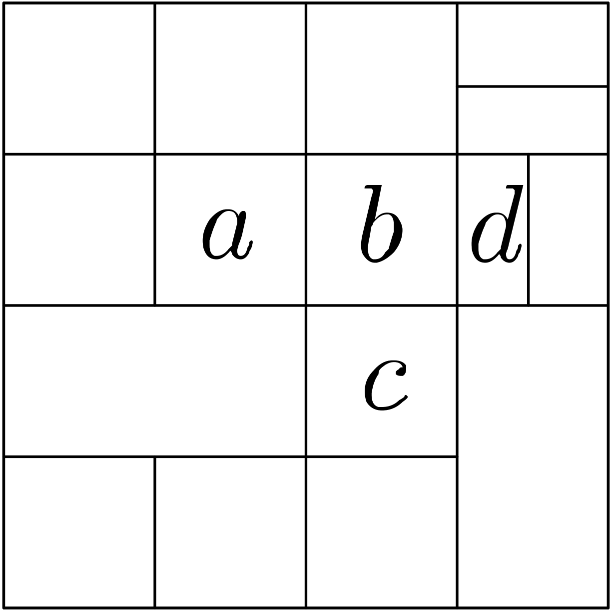
For any sixteen elements $a, b, \ldots$ *in any double semigroup, this equation holds:*

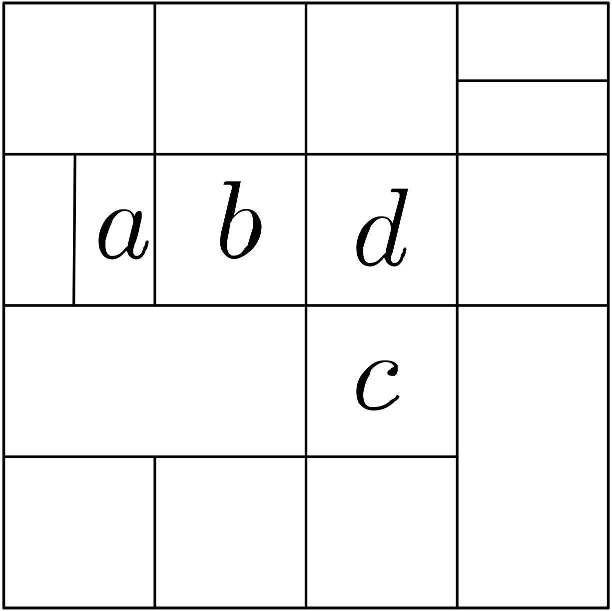
$$\begin{array}{|c|c|c|c|}\hline & & & \\ \hline & a & b & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \quad = \quad \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & b & a & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$$

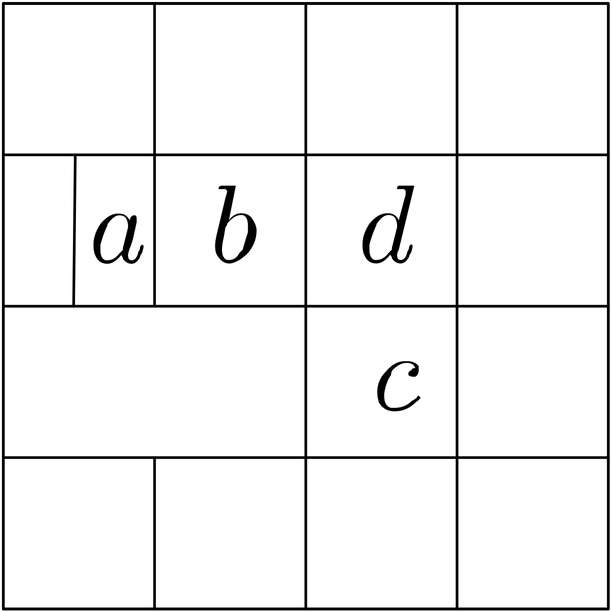
(The empty boxes represent fourteen nameless elements, that are the same on each side of the equation, and in the same order.)

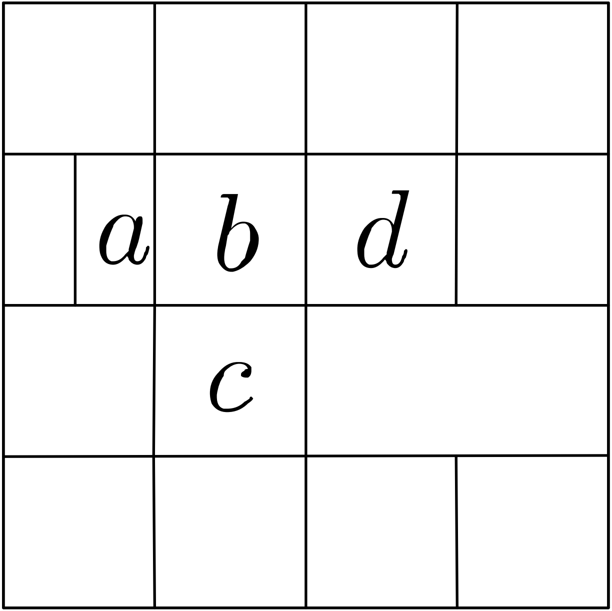
	a	b	
	c	d	

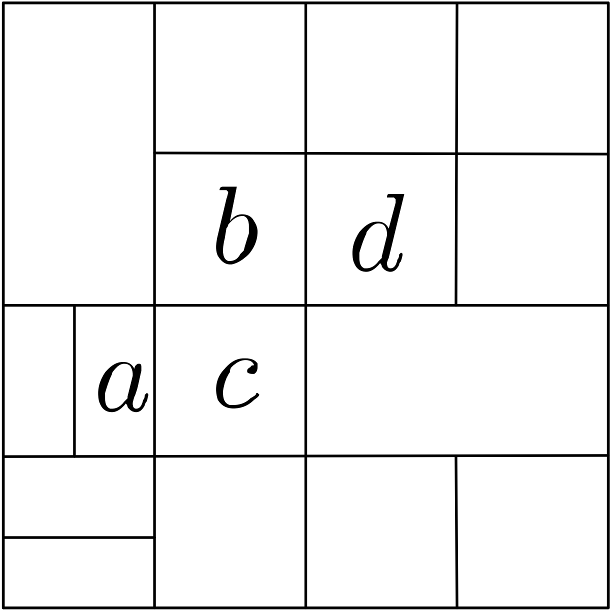

a
b
c
d

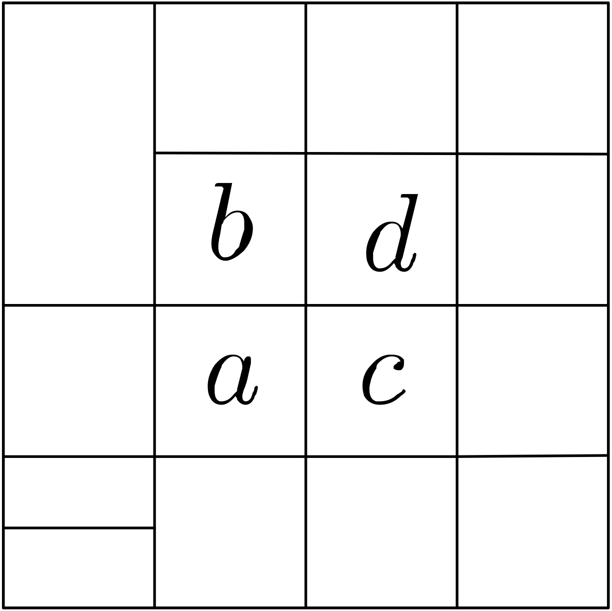

a
b
d
c


a
b
d
c

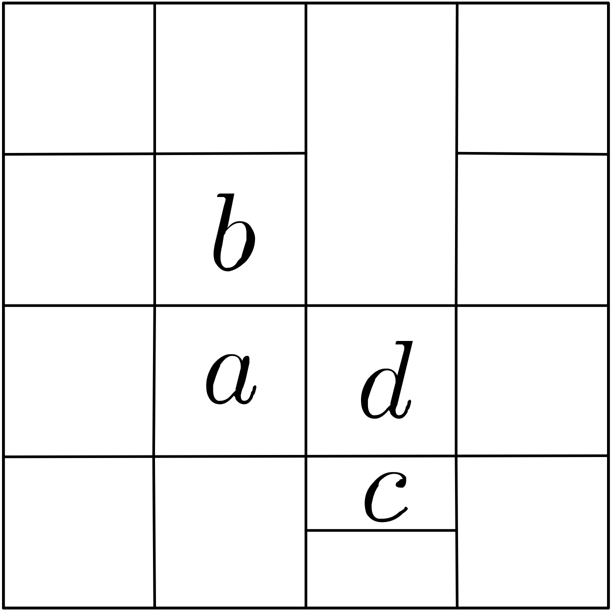

a
b
d
c

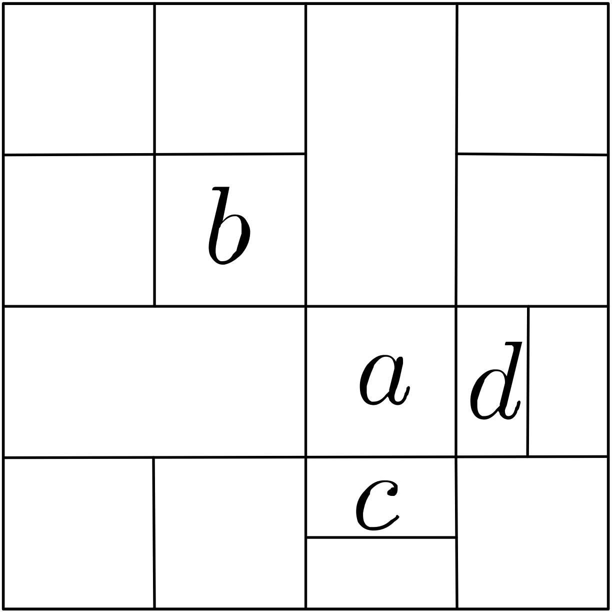

a
b
d
c

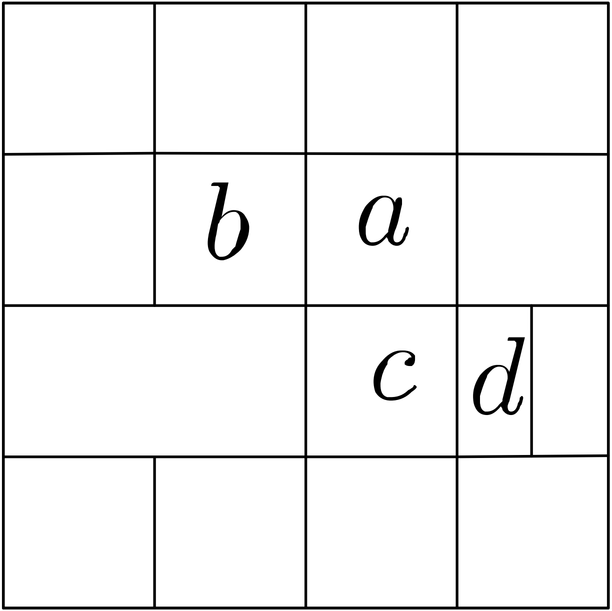

b
d
a
c


b
d
a
c

	b	d	
	a	c	


b
a
d
c


b
a
d
c


b
a
c
d

	b	a	
	c	d	

Double Cancellative Semigroups

Definition

A semigroup S is said to be

- *right cancellative* if, for any $a, b, c \in S$,

$$ac = bc \text{ implies } a = b.$$

- *left cancellative* if, for any $a, b, c \in S$,

$$ca = cb \text{ implies } a = b.$$

- *cancellative* if both left cancellative and right cancellative.

A double semigroup is said to be cancellative if both of its operations are.

Corollary

A cancellative double semigroup D is commutative.

Proof.

Suppose that $a, b \in D$. Let $c \in D$ be any element of D. Then by Theorem 4,

c	c	c	c
c	a	b	c
c	c	c	c
c	c	c	c

=

c	c	c	c
c	b	a	c
c	c	c	c
c	c	c	c

and thus, by the definition of cancellative,

a	b

=

b	a

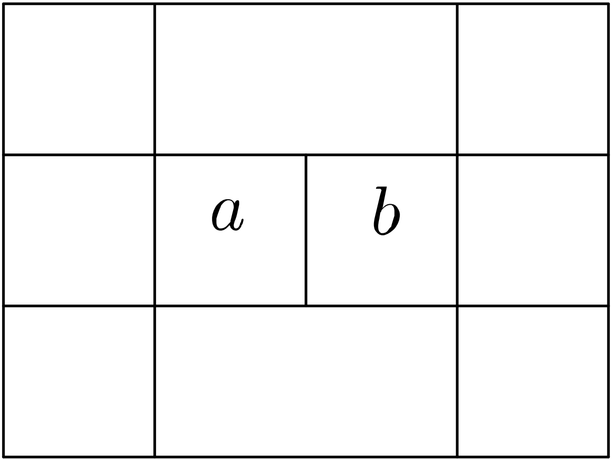
□

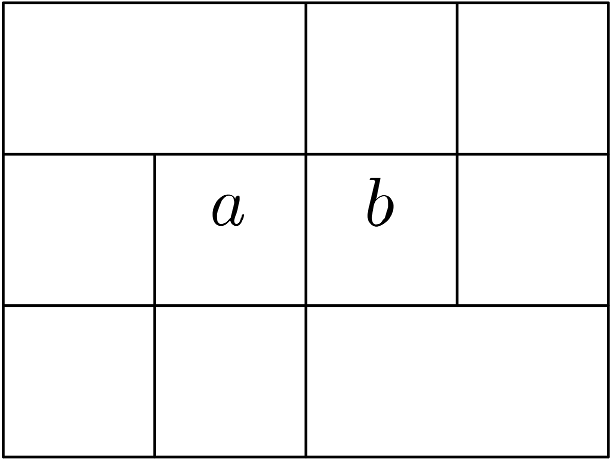
Proposition

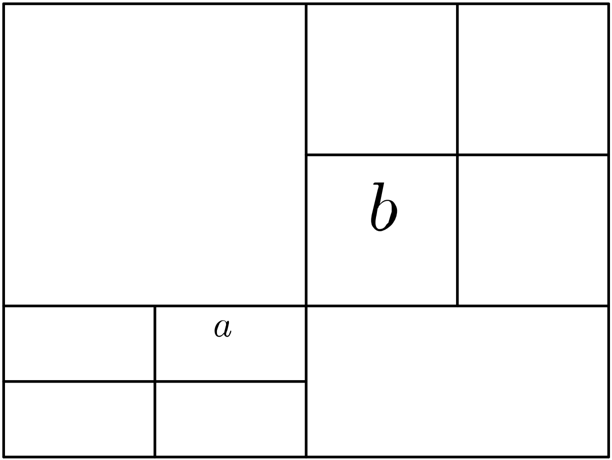
If $(S, \odot, \circledcirc)$ *is a double cancellative semigroup, then* $\odot = \circledcirc$.

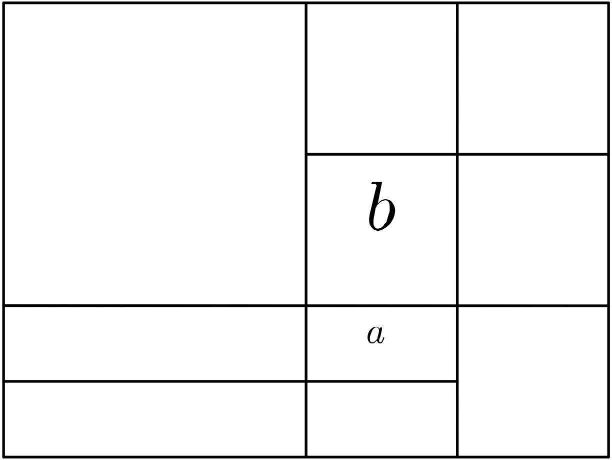
Proof.

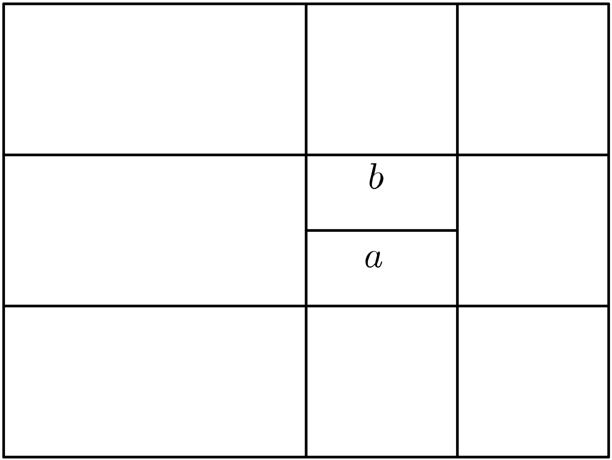
Let $a, b \in S$ and consider the following sequence of tile slidings, where each blank square is some nameless semigroup element:


a
b


a
b


b
a


b
a


b
a

Definition

Two elements x and y in a semigroup S are said to be *inverse* if

$$x = xyx \text{ and } y = yxy.$$

- A semigroup is said to be an *inverse semigroup* if every element has a unique inverse.
- A double semigroup is said to be inverse if both of its operations are.

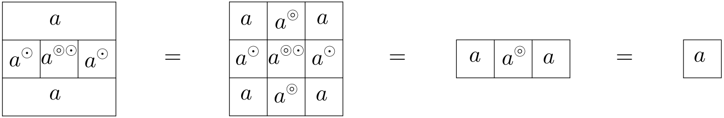
Theorem (Kock)

Double inverse semigroups are commutative.

Need a lemma to prove this:

Lemma

Let S be a double inverse semigroup. Then the inverse operations of S commute. That is, $a^{\odot\circledcirc} = a^{\circledcirc\odot}$ for all $a \in S$.



$$a^{\odot} = a^{\odot} \circledcirc a^{\circledcirc\odot} \circledcirc a^{\odot}?$$

$$
\begin{array}{|c|c|c|}
\hline
a^{\odot} & a^{\circledcirc\odot} & a^{\odot} \\
\hline
\multicolumn{3}{|c|}{a} \\
\hline
a^{\odot} & a^{hv} & a^{\odot} \\
\hline
\end{array}
\quad = \quad
\begin{array}{|c|c|c|}
\hline
a^{\odot} & a^{\circledcirc\odot} & a^{\odot} \\
\hline
a & a^{\circledcirc} & a \\
\hline
a^{\odot} & a^{\circledcirc\odot} & a^{\odot} \\
\hline
\end{array}
\quad = \quad
\begin{array}{|c|c|c|}
\hline
a^{\odot} & a^{\circledcirc\odot} & a^{\odot} \\
\hline
\end{array}
$$

$a^{\odot} = a^{\odot} \circledcirc a^{\circledcirc\odot} \circledcirc a^{\odot}$? Yes.

Recall:

$$a^{\odot} = a^{\odot} \circledcirc a^{\circledcirc\odot} \circledcirc a^{\odot}$$

In particular, for $a^{\circledcirc}$:

$$a^{\circledcirc\odot} = a^{\circledcirc\odot} \circledcirc a^{\odot} \circledcirc a^{\circledcirc\odot}$$

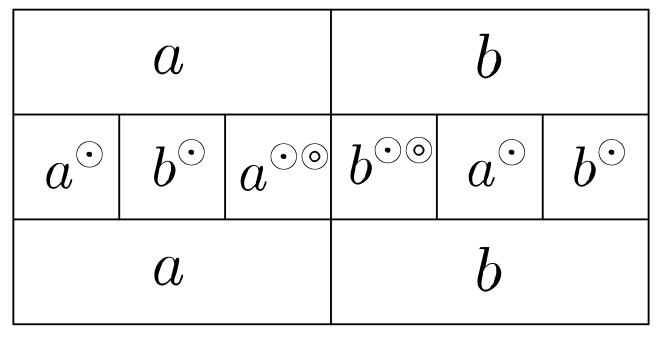
That is,

$$a^{\odot\circledcirc} = a^{\circledcirc\odot}$$

□

Proof of Commutativity.

Fact:



Proof of Commutativity.

Fact:

a	b	$b^{\circledcirc}$	$a^{\circledcirc}$	a	b
$a^{\odot}$	$b^{\odot}$	$a^{\odot\circledcirc}$	$b^{\odot\circledcirc}$	$a^{\odot}$	$b^{\odot}$
a	b	$b^{\circledcirc}$	$a^{\circledcirc}$	a	b

Proof of Commutativity.

Fact:

a	b	$b^{\circledcirc}$	$a^{\circledcirc}$	a	b
$a^{\odot}$	$b^{\odot}$	$a^{\odot\circledcirc}$	$b^{\odot\circledcirc}$	$a^{\odot}$	$b^{\odot}$
a	b	$b^{\circledcirc}$	$a^{\circledcirc}$	a	b
$a^{\odot}$	$b^{\odot}$	$b^{\odot\circledcirc}$	$a^{\odot\circledcirc}$	$a^{\odot}$	$b^{\odot}$
a	b	$b^{\circledcirc}$	$a^{\circledcirc}$	a	b

Proof of Commutativity.

Fact:

a	b	$b^{\circledcirc}$	$a^{\circledcirc}$	a	b
$a^{\odot}$	$b^{\odot}$	$b^{\odot\circledcirc}$	$a^{\odot\circledcirc}$	$a^{\odot}$	$b^{\odot}$
a	b	$b^{\circledcirc}$	$a^{\circledcirc}$	a	b
$a^{\odot}$	$b^{\odot}$	$b^{\odot\circledcirc}$	$a^{\odot\circledcirc}$	$a^{\odot}$	$b^{\odot}$
a	b	$b^{\circledcirc}$	$a^{\circledcirc}$	a	b

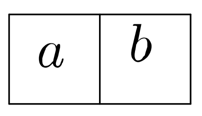
Proof of Commutativity.

Fact:

a	b	$b^{\circledcirc}$	$a^{\circledcirc}$	a	b

Proof of Commutativity.

Fact:



Proof of Commutativity.

Similarly, one calculates that

<table>
<tr><td>$a^{\odot}$</td><td>$b^{\odot}$</td><td>$a^{\circledcirc\odot}$</td><td>$b^{\circledcirc\odot}$</td><td>$a^{\odot}$</td><td>$b^{\odot}$</td></tr>
<tr><td colspan="3">a</td><td colspan="3">b</td></tr>
<tr><td>$a^{\odot}$</td><td>$b^{\odot}$</td><td>$a^{\circledcirc\odot}$</td><td>$b^{\circledcirc\odot}$</td><td>$a^{\odot}$</td><td>$b^{\odot}$</td></tr>
</table>

$=$

$a^{\odot}$	$b^{\odot}$	$a^{\circledcirc\odot}$	$b^{\circledcirc\odot}$	$a^{\odot}$	$b^{\odot}$

The vertical inverse of $a \circledcirc b$ is $a^{\odot} \circledcirc b^{\odot} \circledcirc a^{\circledcirc\odot} \circledcirc b^{\odot\circledcirc} \circledcirc a^{\odot} b^{\odot}$.

Repeat to show:
The vertical inverse of $b \circledcirc a$ is also $a^{\odot} \circledcirc b^{\odot} \circledcirc a^{\circledcirc\odot} \circledcirc b^{\odot\circledcirc} \circledcirc a^{\odot} b^{\odot}$.

This implies:

$$a \circledcirc b = b \circledcirc a$$

□

It can be shown that

Theorem

Double inverse semigroups are essentially commutative inverse semigroups. □