Inverse Restriction Categories and Their Groupoids

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higher-dimensional algebra:

- Ronald Brown: 2-dimensional group theory
- Kock: Double (inverse) semigroups (2007)
- Edmunds: Double magma (2013) and interchange rings (2014)
- Bremner and Madariaga : double semigroups (2014)

Double inverse semigroups: Element-based definition

Definition (Kock, 2007)

A *double inverse semigroup* $(S, \circledcirc, \odot)$ is a set S such that $(S, \circledcirc)$ and $(S, \odot)$ are inverse semigroups and $\circledcirc$ and $\odot$ satisfy the middle-four interchange law:

$$(a \odot b) \circledcirc (c \odot d) = (a \circledcirc c) \odot (b \circledcirc d)$$

Eckmann-Hilton-like result:

Theorem

Double inverse semigroups are improper [DeWolf and Pronk] and commutative [Kock].

Double inverse semigroups: Brown-style definition

Ronald Brown: double groups are single object double groupoids

Definition

A *double inverse semigroup* is a single object double "inverse semigroupoid".

"inverse semigroupoid" is the multiobject version of inverse semigroup
What are these and what do we know about them?

First question: what is an "inverse semigroupoid"?
At each object, we want an inverse semigroup:

- pseudoinverses ($xyx = x$ and $yxy = y$)
- commuting idempotents

What satisfies this need:

inverse restriction categories

Inverse Restriction Categories

Definition (Cockett and Lack, 2002)

A restriction structure on a category $\mathbf{X}$ is an assignment of an arrow $\overline{f_A} : A \to A$ to each arrow $f : A \to B$ in $\mathbf{X}$ satisfying the following four conditions:

(R.1) $f\,\overline{f_A} = f$ for all f

(R.2) $\overline{f_A}\,\overline{g_A} = \overline{g_A}\,\overline{f_A}$ for all $\mathrm{dom}(f) = \mathrm{dom}(g)$

(R.3) $\overline{g_A\,\overline{f_A}} = \overline{g_A}\,\overline{f_A}$ for all $\mathrm{dom}(f) = \mathrm{dom}(g)$

(R.4) $\overline{g_A}\,f = f\,\overline{(gf)_B}$ for all $\mathrm{cod}(f) = \mathrm{dom}(g)$

A category equipped with a restriction structure is called a *restriction category*.

Inverse Restriction Categories

Definition

A restriction category $\mathbf{X}$ is called an *inverse restriction category*, whenever every map f is a restricted isomorphism. That is, each map f has a corresponding map f° such that $f^\circ f = \overline{f}$ and $ff^\circ = \overline{f^\circ}$.

properties:

- existence of pseudoinverses
- idempotents are exactly the restriction idempotents and commute

i.e., these work exactly as desired.

Question: What results in inverse semigroup theory can be extended to inverse restriction categories?
Example: Vagner-Preston works (Cockett and Lack, 2002, Thm 3.8)

Theorem (Ehresmann-Schein-Nambooripad)

The category of inductive groupoids is equivalent to the category of inverse semigroups.

Does this translate to the category of inverse restriction categories, **IRCat**?

Definition

A groupoid $(G, \circ)$ is said to be an *ordered groupoid* whenever there is a partial order $\leq$ on its arrows satisfying the following conditions:

(i) For all arrows $f, g \in G$, $f \leq g$ implies $f^{-1} \leq g^{-1}$.

(ii) For all arrows $a, A, b, B \in G$ such that if $a \leq A$, $b \leq B$ and the composites ab and AB exist, then $ab \leq AB$.

(iii) For all arrows $f : A' \to B$ in G and objects $A \leq A'$ in G, there exists a unique *restriction of f to A* $[f|_* A]$ such that $\mathrm{dom}[f|_* A] = A$ and $[f|_* A] \leq f$.

Definition

An ordered groupoid is said to be

- an *inductive groupoid* whenever its objects form a meet-semilattice,
- a *locally inductive groupoid* whenever there is a partition $\{M_i\}_{i \in I}$ of $\mathbf{G}_0$ into meet-semilattices M_i.

A locally inductive groupoid is said to be *top-heavy* whenever each meet-semilattice M_i admits a top-element $\top_i$.

Notation

Let A be an object of a restriction category $\mathbf{X}$. Let E_A denote the set of restrictions of all endomorphisms on A. That is,

$$E_A = \left\{ \overline{f_A} : A \to A | f : A \to A \in \mathbf{X}_1 \right\}.$$

Proposition

For each object A of a restriction category $\mathbf{X}$, E_A is a meet-semilattice with meets given by $\overline{a} \wedge \overline{b} = \overline{a}\overline{b}$. In addition, E_A has top element 1_A.

Remark

A restriction category is naturally partially equipped with the partial order

$$f \leq g \text{ if and only if } g\overline{f} = f$$

Construction

Given an inverse restriction category $\mathbf{X}$, define a groupoid $\mathcal{G}(\mathbf{X})$:

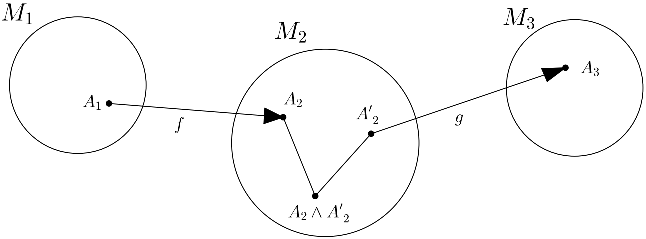
- Objects: $\mathcal{G}(\mathbf{X})_0 = \coprod_{A \in \mathbf{X_0}} E_A$.
- Arrows: $f : A \to B$ in $\mathbf{X}$ corresponds to $f : \overline{f} \to \overline{f^\circ}$ in $\mathcal{G}(\mathbf{X})$.
  - Composition: composition in $\mathbf{X}$
  - Inverses: from restricted isomorphisms in $\mathbf{X}$.

$\mathcal{G}(\mathbf{X})$ is a top-heavy locally inductive groupoid:

- E_A meet semilattice with top 1_A
- partial order from $\mathbf{X}$ – satisfies conditions (i) and (ii)
- restriction : $[f|_*\overline{e}] = f \circ \overline{e}$

Any two maps f and g in an ordered groupoid with $\mathrm{dom} f \wedge \mathrm{cod} g$ existing, there is a tensor product:

$$f \otimes g = [f\,|_*\, \mathrm{dom}(f) \wedge \mathrm{cod}(g)] \bullet [\mathrm{dom}(f) \wedge \mathrm{cod}(g)\, {}_*|\, g]$$



Composition in $\mathbf{X}$ is exactly $\otimes$ in $\mathcal{G}\mathbf{X}$.

Construction

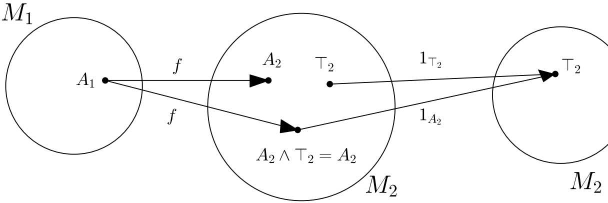
Given a top-heavy locally inductive groupoid $(\mathbf{G}, \bullet, \leq, \{M_i\}_{i\in I})$, define an inverse restriction category $\left(\mathcal{I}(\mathbf{G}), \circ, \overline{(-)}\right)$:

- Objects: The objects are the meet-semilattices M_i.
- Arrows:

$$\mathcal{I}(\mathbf{G})(M_1, M_2) = \{f : A_1 \to A_2 \text{ in } \mathbf{G} \,|\, A_1 \in M_1,\, A_2 \in M_2\}$$

- Composition given by $\otimes$

Identities: Identities on the tops : $1_M = 1_{\top_M}$



$\mathcal{I}(\mathbf{G})$ is an inverse restriction category:

- Restrictions: Given an arrow $f : M_1 \to M_2$ corresponding to an arrow $f : A_1 \to A_2$ in $\mathbf{G}$, define

$$\overline{f} = 1_{A_1} : A_1 \to A_1$$

- Partial Inverses: For each arrow $f : M_1 \to M_2$, define

$$(f^\circ : M_2 \to M_1) = (f^{-1} : A_2 \to A_1)$$

Theorem

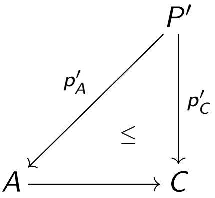
The functors $\mathcal{G}$ and $\mathcal{I}$ form an equivalence

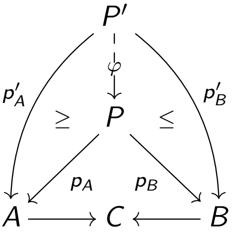
$$\mathbf{IRCat} \underset{\mathcal{I}}{\overset{\mathcal{G}}{\rightleftarrows}} \mathbf{TLIGrpd}$$

To do:
Define: double inverse semigroup is a single-object double inverse restriction category.

Q: What are double restriction categories and how should their restriction structures interact?
A: Unfortunately, no definition yet, but an example.

First, need restricted pullbacks in restriction category $\mathbf{X}$:
Given any cospan $A \longrightarrow C \longleftarrow B$, a restricted pullback is cone consisting of an object P and total arrows $p_{A,B,C} : P \to A/B/C$ satisfying the following universal property:
For each lax cone (P', p'_A, p'_B, p'_C) over $A \longrightarrow B \longleftarrow C$, there is a unique $\varphi : P' \to P$ such that $\varphi \circ p \leq p'$ and $\overline{\varphi} = \overline{p'_A}\,\overline{p'_B}\,\overline{p'_C}$





Let $\mathbf{X}$ be a restriction category. A collection $\mathcal{M}$ of monics in $\mathbf{X}$ is *stable under restricted pullbacks* whenever:

- $\mathcal{M}$ contains all isomorphisms of $\mathcal{M}$,
- $\mathcal{M}$ is closed under composition,
- for each $m : B \to C$ in $\mathcal{M}$ and $f : A \to C$ in $\mathbf{X}$, the restricted pullback

$$\begin{array}{ccc} A \otimes_C B & \xrightarrow{p_2} & B \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle m} \\ A & \xrightarrow[f]{} & C \end{array}$$

of m along f exists and $p_1 \in \mathcal{M}$.

Define a restriction category $\mathrm{Par}(\mathbf{X}, \mathcal{M})$ (Cockett and Lack, 2002) with the following data:

- Objects: Same objects as $\mathbf{X}$
- Arrows: Isomorphism classes of spans

$$X \xleftarrow{\;i\;} D \xrightarrow{\;f\;} Y \ ,$$

with $i \in \mathcal{M}$.

  - Composition: restricted pullback
  - Restriction: $\overline{(i, f)} = (i, i)$

Double Category $\mathbb{P}\mathrm{ar}(\mathbf{X}, \mathcal{M})$

- Objects: Same as $\mathbf{X}$
- Vertical arrows: The total arrows of $\mathbf{X}$
  - total maps form a subcategory so composition is clear
- Horizontal arrows: the arrows of $\mathrm{Par}(\mathbf{X}, \mathcal{M})$
  - composition restricted pullbacks
- Double cells:

$$
\begin{array}{ccccc}
X & \xleftarrow{\;i\;} & D & \xrightarrow{\;f\;} & Y \\
{\scriptstyle u}\bullet\downarrow & \geq & \downarrow{\scriptstyle \alpha} & \leq & \downarrow\bullet{\scriptstyle v} \\
X' & \xleftarrow[\;i'\;]{} & D' & \xrightarrow[\;f'\;]{} & Y'
\end{array}
$$

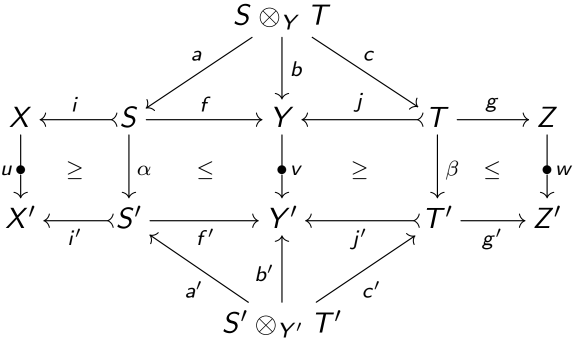
Double Cell Composition

Vertical Composition : compose all arrows vertically – straightforward

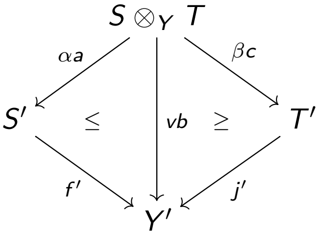
Horizontal Composition: given by universal property of restricted pullback

$$\begin{array}{ccccccccc} X & \overset{i}{\leftarrowtail} & S & \overset{f}{\longrightarrow} & Y & \overset{d}{\leftarrowtail} & T & \overset{x}{\longrightarrow} & Z \\ u\,\downarrow\bullet & \geq & \downarrow\,\alpha & \leq & \bullet\downarrow\, v & \geq & \downarrow\,\beta & \leq & \bullet\downarrow\, w \\ X' & \underset{j}{\leftarrowtail} & S' & \underset{g}{\longrightarrow} & Y' & \underset{c}{\leftarrowtail} & T' & \underset{y}{\longrightarrow} & Z' \end{array}$$

First take the restricted pulbacks:



This gives a lax cone



over

$$S' \xrightarrow{f'} Y' \xleftarrow{j'} T'$$

So there is a unique a unique $\varphi : S \otimes_Y T \to S' \otimes_{Y'} T'$ giving the double cell

$$\begin{array}{ccccc} X & \xleftarrow{\ ia\ } & S \otimes_Y T & \xrightarrow{\ gc\ } & Z \\ {\scriptstyle u}\bullet\downarrow & \geq & \downarrow{\scriptstyle \varphi} & \leq & \downarrow\bullet{\scriptstyle w} \\ X' & \xleftarrow[\ i'a'\ ]{} & S' \otimes_{Y'} T' & \xrightarrow[\ g'c'\ ]{} & Z' \end{array}$$

Vertical Restriction

For each such α, define the vertical restriction $\widetilde{\alpha}$ of α to be

$$\widetilde{\alpha} = \begin{array}{ccccc} X & \overset{i}{\leftarrowtail} & D & \overset{f}{\longrightarrow} & Y \\ {\scriptstyle \overline{u}=1_X}\Big\| & \geq & \Big\downarrow{\scriptstyle \overline{\alpha}} & \leq & \Big\|{\scriptstyle 1_Y=\overline{v}} \\ X & \underset{i}{\leftarrowtail} & D & \underset{f}{\longrightarrow} & Y \end{array}$$

Horizontal Restriction

For each such α, define the horizontal restriction $\overline{\alpha}$ of α to be

$$\overline{\alpha} = \begin{array}{ccccc} X & \overset{i}{\leftarrowtail} & D & \overset{i}{\rightarrowtail} & X \\ {\scriptstyle u}\bullet\!\!\downarrow & \geq & \downarrow{\scriptstyle \alpha} & \leq & \downarrow\!\!\bullet{\scriptstyle u} \\ X' & \underset{j}{\leftarrowtail} & D' & \underset{j}{\rightarrowtail} & X' \end{array}$$

It is quickly seen that the restriction structures commute:

$$
\begin{array}{ccccc}
X & \xleftarrow{\;i\;} & D & \xrightarrow{\;f\;} & Y \\
u \downarrow & \geq & \downarrow \alpha & \leq & \downarrow v \\
X' & \xleftarrow[\;i'\;]{} & D' & \xrightarrow[\;f'\;]{} & Y'
\end{array}
\;\xrightarrow{\widetilde{(-)}}\;
\begin{array}{ccccc}
X & \xleftarrow{\;i\;} & D & \xrightarrow{\;f\;} & Y \\
\| & \geq & \downarrow \overline{\alpha} & \leq & \| \\
X & \xleftarrow[\;i\;]{} & D & \xrightarrow[\;f\;]{} & Y
\end{array}
\;\xrightarrow{\overline{(-)}}\;
\begin{array}{ccccc}
X & \xleftarrow{\;i\;} & D & \xrightarrow{\;i\;} & X \\
\| & \geq & \downarrow \overline{\alpha} & \leq & \| \\
X & \xleftarrow[\;i\;]{} & D & \xrightarrow[\;i\;]{} & X
\end{array}
$$

$$
\begin{array}{ccccc}
X & \xleftarrow{\;i\;} & D & \xrightarrow{\;f\;} & Y \\
u \downarrow & \geq & \downarrow \alpha & \leq & \downarrow v \\
X' & \xleftarrow[\;i'\;]{} & D' & \xrightarrow[\;f'\;]{} & Y'
\end{array}
\;\xrightarrow{\overline{(-)}}\;
\begin{array}{ccccc}
X & \xleftarrow{\;i\;} & D & \xrightarrow{\;i\;} & X \\
u \downarrow & \geq & \downarrow \alpha & \leq & \downarrow u \\
X' & \xleftarrow[\;j\;]{} & D' & \xrightarrow[\;j\;]{} & X'
\end{array}
\;\xrightarrow{\widetilde{(-)}}\;
\begin{array}{ccccc}
X & \xleftarrow{\;i\;} & D & \xrightarrow{\;i\;} & X \\
\| & \geq & \downarrow \overline{\alpha} & \leq & \| \\
X & \xleftarrow[\;i\;]{} & D & \xrightarrow[\;i\;]{} & X
\end{array}
$$

Thank you!