

Restriction Monads

Category Theory 2016

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Restriction Categories (Cockett and Lack, 2002)

A category \mathbf{X} is called a restriction category when it can be equipped with an assignment

$$(f : A \rightarrow B) \mapsto (\overline{f}_A : A \rightarrow A)$$

of all arrows f in \mathbf{X} to an endomorphism \overline{f} satisfying:

1. For all maps f , $f \overline{f}_A = f$.
2. For all maps $f : A \rightarrow B$ and $g : A \rightarrow B'$, $\overline{f}_A \overline{g}_A = \overline{g}_A \overline{f}_A$.
3. For all maps $f : A \rightarrow B$ and $g : A \rightarrow B'$, $\overline{g}_A \overline{f}_A = \overline{g}_A \overline{f}_A$.
4. For all maps $f : B \rightarrow A$ and $g : A \rightarrow B'$, $\overline{g}_A f = f \overline{(gf)}_B$.

Restriction Category Objects

- ▶ Obvious data needed:

$$\mathbb{C} = \mathbf{X}_1 \begin{matrix} t \\ \times \\ s \end{matrix} \mathbf{X}_1 \xrightarrow{c} \mathbf{X}_1 \begin{matrix} \xrightarrow{s} \\ \xleftarrow{u} \\ \xrightarrow{t} \end{matrix} \mathbf{X}_0$$

- ▶ What additional data is needed to allow us to diagrammatically express (R.1) - (R.4)?
- ▶ Also, want to keep an eye out and avoid using the fact that restriction categories are internal to **Set**.

Restriction Monads

In a bicategory, a restriction monad consists of a 0-cell x , 1-cells

$$T, D, E : x \rightarrow x$$

and 2-cells

- ▶ $\eta : 1_T \Rightarrow T$,
- ▶ $\mu : T^2 \Rightarrow T$,
- ▶ $[\mu |_* DE] : DE \Rightarrow D$,
- ▶ $\rho : D \Rightarrow E$ (epic),
- ▶ $\iota : E \Rightarrow T$ (monic),
- ▶ $\Delta : T \Rightarrow TD$,
- ▶ $\tau : D^2 \Rightarrow D^2$ and
- ▶ $\psi : DT \Rightarrow TD$

satisfying conditions corresponding to (R.1) through (R.4) plus the usual monad laws.

Example: $\text{Par} : \mathbf{Set} \rightarrow \mathbf{Set}$ in \mathbf{Cat}

Define a functor $\text{Par} : \mathbf{Set} \rightarrow \mathbf{Set}$ by $\text{Par}(A) = A \amalg \{\star\}$ and

$$\text{Par}(f : A \rightarrow B)(x) = \begin{cases} f(x) & x \in A \\ \star & x = \star \end{cases}$$

A monad with

- ▶ $\eta_A : A \rightarrow A \amalg \{\star\} : a \mapsto a$
- ▶ $\mu_A : (A \amalg \{\star\}) \amalg \{\star\} \rightarrow A \amalg \{\star\}$

Its Kleisli arrows are total representations of partial functions; a partial function $f : A \rightarrow B$ can be thought of as

$$f : A \rightarrow B \amalg \{\star\}$$

Example: $\text{Par} : \mathbf{Set} \rightarrow \mathbf{Set}$ in \mathbf{Cat}

Giving Par a restriction monad structure: $\text{Set } E = D = \text{Par}$.

(R.1): “ $f = f\bar{f}$ ”

$$\begin{array}{ccc}
 \text{Par} & \xrightarrow{\Delta} & \text{Par}^2 \\
 \parallel \scriptstyle 1_{\text{Par}} & & \downarrow \scriptstyle \text{Par } \rho \\
 \text{Par} & \xleftarrow{\mu \cdot \text{Par } \iota} & \text{Par}^2
 \end{array}
 \quad \xrightarrow{A} \quad
 \begin{array}{ccc}
 A \amalg \{\star\} & \xrightarrow{\Delta_A} & (A \amalg \{\star\}) \amalg \{\star\} \\
 \parallel \scriptstyle 1_{\text{Par}(A)} & & \downarrow \scriptstyle \text{Par}(A) \rho_A \\
 A \amalg \{\star\} & \xleftarrow{\mu \cdot \text{Par}(A) \iota_A} & (A \amalg \{\star\}) \amalg \{\star\}
 \end{array}$$

Implies that $\rho = 1$.

Example: $R(\mathbf{X}) : \mathbf{X}_0 \rightarrow \mathbf{X}_0$ in $\mathbf{Span}(\mathbf{Set})$

Let \mathbf{X} be a small restriction category.

$$T = \begin{array}{ccc} & \mathbf{X}_1 & \\ s \swarrow & & \searrow t \\ \mathbf{X}_0 & & \mathbf{X}_0 \end{array}$$

$$\eta : 1_T \Rightarrow T : \mathbf{X}_0 \rightarrow \mathbf{X}_1 : A \mapsto 1_A$$

$$\begin{array}{ccccc} & & \mathbf{X}_0 & & \\ & 1 \swarrow & \downarrow \eta & \searrow 1 & \\ \mathbf{X}_0 & & & & \mathbf{X}_0 \\ & s \swarrow & & \searrow t & \\ & & \mathbf{X}_1 & & \end{array}$$

$$\mu : T^2 \Rightarrow T : \mathbb{C} \rightarrow \mathbf{X}_1 : (f, g) \mapsto gf$$

$$\begin{array}{ccccc} & & \mathbb{C} & & \\ & s\pi_1 \swarrow & \downarrow \mu & \searrow t\pi_2 & \\ \mathbf{X}_0 & & & & \mathbf{X}_0 \\ & s \swarrow & & \searrow t & \\ & & \mathbf{X}_1 & & \end{array}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \rightarrow \mathbf{X}_0$ in $\mathbf{Span}(\mathbf{Set})$

$$D = \begin{array}{ccc} & \mathbf{X}_1 & \\ s \swarrow & & \searrow s \\ \mathbf{X}_0 & & \mathbf{X}_0 \end{array}$$

$$\Delta : T \Rightarrow TD : \mathbf{X}_1 \rightarrow \mathbb{D} : f \mapsto (f, f)$$

$$\begin{array}{ccccc} & & \mathbf{X}_1 & & \\ & s \swarrow & & \searrow t & \\ \mathbf{X}_0 & & & & \mathbf{X}_0 \\ & s\pi_1 \swarrow & \mathbb{D} & \searrow t\pi_2 & \\ & & & & \end{array}$$

Δ

$$\mathbb{D} = \{(f, g) \in \mathbf{X}_1 \times \mathbf{X}_1 : sf = sg\}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \rightarrow \mathbf{X}_0$ in $\mathbf{Span}(\mathbf{Set})$

$$E = \begin{array}{ccc} & \bar{\mathbf{X}}_1 & \\ s \swarrow & & \searrow t \\ \mathbf{X}_0 & & \mathbf{X}_0 \end{array}$$

where

$$\bar{\mathbf{X}}_1 = \{\bar{f} : f \in \mathbf{X}\}$$

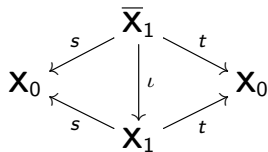
And define

$$\rho : D \Rightarrow E : \mathbf{X}_1 \rightarrow \bar{\mathbf{X}}_1 : f \mapsto \bar{f}$$

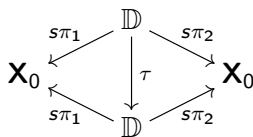
$$\begin{array}{ccccc} & & \mathbf{X}_1 & & \\ & s \swarrow & & \searrow s & \\ \mathbf{X}_0 & & & & \mathbf{X}_0 \\ & s \swarrow & \downarrow \rho & \searrow t & \\ & & \bar{\mathbf{X}}_1 & & \end{array}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \rightarrow \mathbf{X}_0$ in $\mathbf{Span}(\mathbf{Set})$

$$\iota : E \Rightarrow T : \bar{\mathbf{X}}_1 \rightarrow \mathbf{X}_1 : \bar{f} \mapsto \bar{f}$$

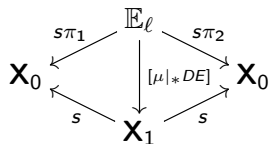


$$\tau : D^2 \Rightarrow D^2 : \mathbb{D} \rightarrow \mathbb{D} : (f, g) \mapsto (g, f)$$

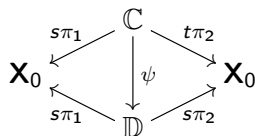


Example: $R(\mathbf{X}) : \mathbf{X}_0 \rightarrow \mathbf{X}_0$ in $\mathbf{Span}(\mathbf{Set})$

$$[\mu |_* DE] : DE \Rightarrow D : \mathbb{E}_\ell \rightarrow \mathbf{X}_1 : (\bar{f}, g) \mapsto g\bar{f}$$



$$\psi : DT \Rightarrow TD : \mathbb{C} \rightarrow \mathbb{D} : (f, g) \mapsto (gf, f)$$



Example: $R(\mathbf{X}) : \mathbf{X}_0 \rightarrow \mathbf{X}_0$ in $\mathbf{Span}(\mathbf{Set})$

(R.1): “ $f = f\bar{f}$ ”

$$\begin{array}{ccc}
 T & \xrightarrow{\Delta} & TD \\
 \parallel^{1_T} & & \downarrow T\rho \\
 T & \xleftarrow{\mu \cdot T\iota} & TE
 \end{array}$$

$$f \mapsto (f, f) \mapsto (\bar{f}, f) \mapsto f\bar{f}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \rightarrow \mathbf{X}_0$ in $\mathbf{Span}(\mathbf{Set})$

(R.2): " $\bar{f} \bar{g} = \bar{g} \bar{f}$ "

$$\begin{array}{ccc}
 D^2 & \xrightarrow{\rho^2} & E^2 \\
 \downarrow \tau & & \searrow \mu \cdot \iota^2 \\
 D^2 & \xrightarrow{\rho^2} & E^2 \\
 & & \nearrow \mu \cdot \iota^2 \\
 & & T
 \end{array}$$

$$(f, g) \mapsto (\bar{f}, \bar{g}) \mapsto \bar{g} \bar{f}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \rightarrow \mathbf{X}_0$ in $\mathbf{Span}(\mathbf{Set})$

$$(R.3): \overline{gf} = \overline{g} \overline{f}$$

$$\begin{array}{ccc}
 D^2 & \xrightarrow{D\rho} & DE \\
 \rho^2 \downarrow & & \searrow [\mu |_* DE] \\
 E^2 & & D \\
 \mu.\iota^2 \downarrow & & \swarrow \rho \\
 T & \xleftarrow{\iota} & E
 \end{array}$$

$$(f, g) \mapsto (\overline{f}, g) \mapsto g\overline{f} \mapsto \overline{g\overline{f}}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \rightarrow \mathbf{X}_0$ in $\mathbf{Span}(\mathbf{Set})$

(R.4): " $\overline{gf} = f\overline{gf}$ "

$$\begin{array}{ccccc}
 DT & \xrightarrow{\rho T} & ET & \xrightarrow{\mu \cdot \iota T} & T \\
 \downarrow \psi & & & & \uparrow \mu \cdot T \iota \\
 TD & \xrightarrow{T\rho} & & & TE
 \end{array}$$

$$(f, g) \mapsto (gf, f) \mapsto (\overline{gf}, f) \mapsto f\overline{gf}$$

Restriction Category Objects

A restriction category in \mathbf{C} (a category with pullbacks over s and t) contains the following data:

$$\mathbb{C} \xrightarrow{c} \mathbf{X}_1 \begin{array}{c} \xrightarrow{s} \mathbf{X}_0 \\ \xleftarrow{u} \mathbf{X}_0 \\ \xrightarrow{t} \mathbf{X}_0 \end{array} \quad \text{with a curved arrow } r: \mathbf{X}_1 \rightarrow \mathbf{X}_1$$

\mathbb{C} and \mathbb{D} are defined by the pullback squares

$$\begin{array}{ccc} & \mathbb{C} & \\ \swarrow & & \searrow \\ \mathbf{X}_1 & & \mathbf{X}_1 \\ \searrow & & \swarrow \\ & \mathbf{X}_0 & \end{array} \quad \text{and} \quad \begin{array}{ccc} & \mathbb{D} & \\ \swarrow & & \searrow \\ \mathbf{X}_1 & & \mathbf{X}_1 \\ \searrow & & \swarrow \\ & \mathbf{X}_0 & \end{array}$$

Satisfying the usual category axioms and

$$sr = s = tr$$

Restriction Category Objects: (R.1) – (R.4)

$$\begin{array}{ccc}
 \mathbf{X}_1 & \xrightarrow{\Delta} & \mathbb{D} \\
 \downarrow 1 & & \downarrow r \times 1 \\
 \mathbf{X}_1 & \xleftarrow{c} & \mathbb{C}
 \end{array}$$

$$\begin{array}{ccc}
 \mathbb{D} & \xrightarrow{\tau} & \mathbb{D} \\
 \downarrow r^2 & & \downarrow r^2 \\
 \mathbb{C} & & \mathbb{C} \\
 \searrow c & & \swarrow c \\
 & \mathbf{X}_1 &
 \end{array}$$

$$\begin{array}{ccc}
 \mathbb{D} & \xrightarrow{r^2} & \mathbb{C} \\
 \downarrow r \times 1 & & \downarrow c \\
 \mathbb{C} & & \mathbf{X}_1 \\
 \searrow c & & \nearrow r \\
 & \mathbf{X}_1 &
 \end{array}$$

$$\begin{array}{ccc}
 \mathbb{C} & \xrightarrow{\psi} & \mathbb{D} \\
 \downarrow 1 \times r & & \downarrow r \times 1 \\
 \mathbb{C} & & \mathbb{C} \\
 \searrow c & & \swarrow c \\
 & \mathbf{X}_1 &
 \end{array}$$

Restriction Category Objects

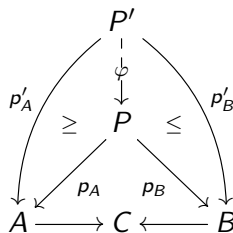
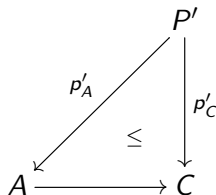
Definition

A double restriction category is a restriction category internal to **rCat**.

Restricted Pullbacks

Given any cospan $A \longrightarrow C \longleftarrow B$, a restricted pullback is cone consisting of an object P and total arrows $p_{A,B,C} : P \rightarrow A/B/C$ satisfying the following universal property:

For each lax cone (P', p'_A, p'_B, p'_C) over $A \longrightarrow B \longleftarrow C$, there is a unique $\varphi : P' \rightarrow P$ such that $\varphi \circ p' \leq p$ and $\overline{\varphi} = \overline{p'_A} \overline{p'_B} \overline{p'_C}$



Let \mathbf{X} be a restriction category. A collection \mathcal{M} of monics in \mathbf{X} is *stable under restricted pullbacks* whenever:

- ▶ \mathcal{M} contains all isomorphisms of \mathbf{X} ,
- ▶ \mathcal{M} is closed under composition,
- ▶ for each $m : B \rightarrow C$ in \mathcal{M} and $f : A \rightarrow C$ in \mathbf{X} , the restricted pullback

$$\begin{array}{ccc}
 A \otimes_C B & \xrightarrow{p_2} & B \\
 \downarrow p_1 & & \downarrow m \\
 A & \xrightarrow{f} & C
 \end{array}$$

of m along f exists and $p_1 \in \mathcal{M}$.

Define a restriction category $\text{Par}(\mathbf{X}, \mathcal{M})$ (Cockett and Lack, 2002) with the following data:

- ▶ Objects: Same objects as \mathbf{X}
- ▶ Arrows: Isomorphism classes of spans

$$X \longleftarrow^i D \longrightarrow^f Y ,$$

with $i \in \mathcal{M}$.

- ▶ Composition: restricted pullback
- ▶ Restriction: $\overline{(i, f)} = (i, i)$

Example: Double Category $\mathbb{P}\text{ar}(\mathbf{X}, \mathcal{M})$

- ▶ Objects: Same as \mathbf{X}
- ▶ Vertical arrows: The total arrows of \mathbf{X}
 - ▶ total maps form a subcategory so composition is inherited from \mathbf{X} .
- ▶ Horizontal arrows: the arrows of $\mathbb{P}\text{ar}(\mathbf{X}, \mathcal{M})$
 - ▶ composition by restricted pullbacks
- ▶ Double cells:

$$\begin{array}{ccccc}
 X & \xleftarrow{i} & D & \xrightarrow{f} & Y \\
 \downarrow u & \geq & \downarrow \alpha & \leq & \downarrow v \\
 X' & \xleftarrow{i'} & D' & \xrightarrow{f'} & Y'
 \end{array}$$

Double Cell Composition

Vertical Composition : compose all arrows vertically – straightforward

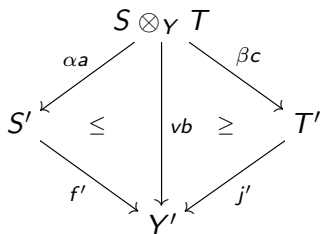
Horizontal Composition: given by universal property of restricted pullback

$$\begin{array}{ccccccc}
 X & \xleftarrow{i} & S & \xrightarrow{f} & Y & \xleftarrow{d} & T & \xrightarrow{x} & Z \\
 \downarrow u \bullet & \geq & \downarrow \alpha & \leq & \downarrow v \bullet & \geq & \downarrow \beta & \leq & \downarrow w \bullet \\
 X' & \xleftarrow{j} & S' & \xrightarrow{g} & Y' & \xleftarrow{c} & T' & \xrightarrow{y} & Z'
 \end{array}$$

First take the restricted pulbacks:

$$\begin{array}{ccccccc}
 & & S \otimes_Y T & & & & \\
 & & \swarrow a & \downarrow b & \searrow c & & \\
 X & \xleftarrow{i} & S & \xrightarrow{f} & Y & \xleftarrow{j} & T \xrightarrow{g} Z \\
 \downarrow u & \geq & \downarrow \alpha & \leq & \downarrow v & \geq & \downarrow \beta \leq \downarrow w \\
 X' & \xleftarrow{i'} & S' & \xrightarrow{f'} & Y' & \xleftarrow{j'} & T' \xrightarrow{g'} Z' \\
 & & \swarrow a' & \uparrow b' & \searrow c' & & \\
 & & S' \otimes_{Y'} T' & & & &
 \end{array}$$

This gives a lax cone



over

$$S' \xrightarrow{f'} Y' \xleftarrow{j'} T'$$

So there is a unique $\varphi : S \otimes_Y T \rightarrow S' \otimes_{Y'} T'$ giving the double cell

$$\begin{array}{ccccc}
 X & \xleftarrow{ia} & S \otimes_Y T & \xrightarrow{gc} & Z \\
 \downarrow u \bullet & \geq & \downarrow \varphi & \leq & \downarrow \bullet w \\
 X' & \xleftarrow{i'a'} & S' \otimes_{Y'} T' & \xrightarrow{g'c'} & Z'
 \end{array}$$

Vertical Restriction

For each such α , define the vertical restriction $\tilde{\alpha}$ of α to be

$$\tilde{\alpha} = \begin{array}{ccccc} X & \xleftarrow{i} & D & \xrightarrow{f} & Y \\ \parallel & \geq & \downarrow \bar{\alpha} & \leq & \parallel \\ X & \xleftarrow{i} & D & \xrightarrow{f} & Y \end{array} \begin{array}{l} \bar{u}=1_X \\ \\ \\ \\ 1_Y=\bar{v} \end{array}$$

Horizontal Restriction

For each such α , define the horizontal restriction $\bar{\alpha}$ of α to be

$$\bar{\alpha} = \begin{array}{ccccc} X & \xleftarrow{i} & D & \xrightarrow{i} & X \\ \downarrow u \bullet & \geq & \downarrow \alpha & \leq & \bullet u \downarrow \\ X' & \xleftarrow{j} & D' & \xrightarrow{j} & X' \end{array}$$

It is quickly seen that the restriction structures commute:

$$\begin{array}{ccc}
 \begin{array}{c} X \xleftarrow{i} \langle D \rangle \xrightarrow{f} Y \\ u \bullet \quad \geq \quad \downarrow \alpha \leq \quad \bullet v \\ \downarrow \quad \quad \quad \downarrow \\ X' \xleftarrow{i'} \langle D' \rangle \xrightarrow{f'} Y' \end{array} & \xrightarrow{(\widetilde{-})} & \begin{array}{c} X \xleftarrow{i} \langle D \rangle \xrightarrow{f} Y \\ \parallel \quad \geq \quad \downarrow \bar{\alpha} \leq \quad \parallel \\ X \xleftarrow{i} \langle D \rangle \xrightarrow{f} Y \end{array} & \xrightarrow{(\overline{-})} & \begin{array}{c} X \xleftarrow{i} \langle D \rangle \xrightarrow{i} X \\ \parallel \quad \geq \quad \downarrow \bar{\alpha} \leq \quad \parallel \\ X \xleftarrow{i} \langle D \rangle \xrightarrow{i} X \end{array} \\
 \\
 \begin{array}{c} X \xleftarrow{i} \langle D \rangle \xrightarrow{f} Y \\ u \bullet \quad \geq \quad \downarrow \alpha \leq \quad \bullet v \\ \downarrow \quad \quad \quad \downarrow \\ X' \xleftarrow{i'} \langle D' \rangle \xrightarrow{f'} Y' \end{array} & \xrightarrow{(\overline{-})} & \begin{array}{c} X \xleftarrow{i} \langle D \rangle \xrightarrow{i} X \\ u \bullet \quad \geq \quad \downarrow \alpha \leq \quad \bullet u \\ \downarrow \quad \quad \quad \downarrow \\ X' \xleftarrow{j} \langle D' \rangle \xrightarrow{j} X' \end{array} & \xrightarrow{(\widetilde{-})} & \begin{array}{c} X \xleftarrow{i} \langle D \rangle \xrightarrow{i} X \\ \parallel \quad \geq \quad \downarrow \bar{\alpha} \leq \quad \parallel \\ X \xleftarrow{i} \langle D \rangle \xrightarrow{i} X \end{array}
 \end{array}$$